1. Preliminaries

This chapter recalls some useful definitions and notations that will be used throughout the entire course.

1.1 Sets

Notation 1.1. *Let A be a set. We denote the statement "an element x is in A" by $x \in A$.*

Notation 1.2. *The empty set is denoted $\varnothing$.*

Definition 1.1.1

Let A and B be two sets then

- $A \subset B$ means "the set A is included in the set B" or "B includes A" or "A is a subset of B" (see left of Figure 1.1),
- A and B are said to be equal if $A \subset B$ and $B \subset A$, it is denoted $A = B$.

Definition 1.1.2

Let A and B be two sets then

- the set C is the **intersection** of A and B, denoted $C = A \cap B$ if $C = \{x / x \in A \quad \text{and} \quad x \in B\}$ (see center of Figure 1.1),
- the set C is the **union** of A and B, denoted $C = A \cup B$ if $C = \{x / x \in A \quad \text{or} \quad x \in B\}$ (see right of Figure 1.1),

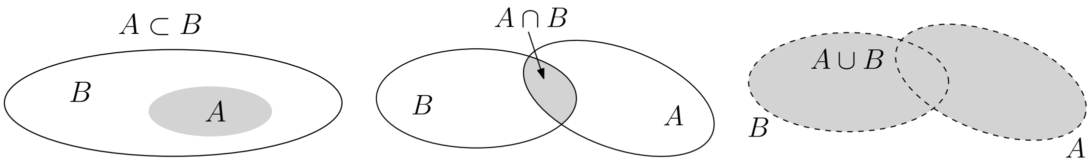
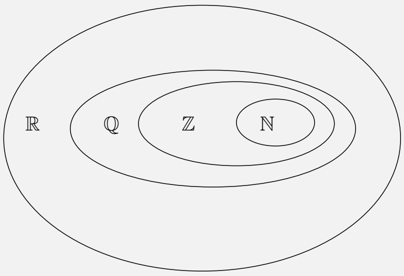


Figure 1.1: Relationships between sets: inclusion in left, intersection in center and union in right.

■ **Example 1.1**
usual sets:

- $\mathbb{N}$: set of natural numbers: $\{1,2,3,\dots\}$,
- $\mathbb{Z}$: set of integers: $\{\dots,-3,-2,-1,0,1,2,3,\dots\}$,
- $\mathbb{Q}$: set of rational numbers $\frac{p}{q}$ where $p,q \in \mathbb{Z}$ (e.g $5/3, -7/2, \dots$),
- $\mathbb{R}$: set real numbers.



We have the following property:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Definition 1.1.3
Union and intersection of an arbitrary collection of sets ($\mathscr{F}$ will denote the collection):

$$\bigcup_{A \in \mathscr{F}} A = \{x/x \in A \text{ for some } A \in \mathscr{F}\}$$

$$\bigcap_{A \in \mathscr{F}} A = \{x/x \in A \text{ for each } A \in \mathscr{F}\}$$

In particular, if we consider a collection of a finite number of sets:

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{k=1}^{n} A_k,$$

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{k=1}^{n} A_k.$$

For an infinite, but countable, collection of sets, we use the notation: $\bigcup_{k=1}^{\infty} A_k$ and $\bigcap_{k=1}^{\infty} A_k$.

1.2 Quantifiers

Definition 1.2.1
Quantifiers notations:

- $\forall$ means "for all" or "for any" or "for an arbitrary",
- $\exists$ means "there exists",
- $\exists!$ means "there exists a unique"

■ **Example 1.2**
Let D be a set,

- $\forall x \in D$ means "for all $x \in D$"
- $\exists x \in D$ means "there exists an $x \in D$"
- $\exists! x \in D$ means "there exists a unique $x \in D$"

R Be careful about the order! Quantifiers do not necessarily commute: if we write $A(x,y)$ a statement that depends on two variables x and y, we have:

- $\forall x, \forall y, A(x,y) = \forall y, \forall x, A(x,y)$
- $\exists x, \exists y, A(x,y) = \exists y, \exists x, A(x,y)$
- $\forall x, \exists y, A(x,y) \neq \exists y, \forall x, A(x,y)$

1.3 A bit of logic

Let A and B be two statements (e.g $A =$"x is positive"). A statement can be either **true** or **false**.

Definition 1.3.1
The statement C defined by

- $C = A \wedge B$ means C is true if both A **and** B are true,
- $C = A \vee B$ means C is true if either A **or** B, or both, are true,
- $C = \neg A = \bar{A}$ means "**not** A", i.e C is true if A is false and C is false if A is true.

Some properties:

$$\overline{A \wedge B} = \bar{A} \vee \bar{B},$$
$$\overline{A \vee B} = \bar{A} \wedge \bar{B}.$$

Theorem 1.3.1 — De Morgan laws
Let $A(x)$ a statement depending on the quantity x, we have

$$\overline{\exists x, A(x)} = \forall x, \overline{A(x)}$$

$$\overline{\forall x, A(x)} = \exists x, \overline{A(x)}$$

Example 1.3

$$\overline{\exists x, f(x) = 0} = \forall x, f(x) \neq 0$$
$$\overline{\forall x, f(x) = 0} = \exists x, f(x) \neq 0$$
$$\overline{\forall x, \exists N, \forall n \geq N, P(x,n)} = \exists x, \forall N, \exists n \geq N, \overline{P(x,n)}$$

Definition 1.3.2
Let A and B be two statements, we have

- $A \Rightarrow B$ means "A implies B" or equivalently "if A is true then B is also true",
  Example: $x > 1 \Rightarrow x > 0$
- $A \Leftrightarrow B$ means "A is equivalent to B" or "$(A \Rightarrow B) \wedge (B \Rightarrow A)$",
  Example: $x > 0 \Leftrightarrow x^3 > 0$

Some properties:

- $\overline{A \Rightarrow B} = A \wedge \bar{B}$,
- ☠ $\overline{A \Rightarrow B} \neq \bar{A} \Rightarrow \bar{B}$!!!
  Example: $x \leq 1 \not\Rightarrow x \leq 0$, you can pick up any number between 0 and 1
- Contraposition: $(A \Rightarrow B) \Leftrightarrow (\bar{B} \Rightarrow \bar{A})$
  Example: $(x > 1 \Rightarrow x > 0) \Leftrightarrow (\overline{x > 0} \Rightarrow \overline{x > 1}) = (x \leq 0 \Rightarrow x \leq 1)$

1.4 Functions

Definition 1.4.1
A function from a set X to a set Y is a rule that assigns to each element of X an element of Y. We denote

$$f : X \to Y$$

The sets X and Y are called "**domain** of f" and "**codomain** of f", respectively. For $x \in X$, we denote $y = f(x), y \in Y$. The **range** of f is the set

$$\{y \in Y / \exists x \in X \text{ such that } y = f(x)\}.$$

R In this course, we will deal only with real functions of a real variable. We generally denote

$$f : D \subset \mathbb{R} \to \mathbb{R}$$

and the range of f is denoted $f(D)$.

Definition 1.4.2
The graph of f is the set of points in the Cartesian plane of coordinates $(x, f(x))$.

Proposition 1.4.1 — Basic operation on functions.
Assume that the function f and g are both defined on a domain D, then $\forall x \in D$,
linearity $(f+g)(x) = f(x) + g(x)$,
product $(fg)(x) = f(x)g(x)$,
quotient $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if $g(x) \neq 0$.

Definition 1.4.3 — Composition of functions.
If $\forall x \in D, g(x)$ is in the domain of f, then the composition of f and g is defined by $f \circ g$ ("f composed with g"):

$$\forall x \in D, (f \circ g)(x) = f(g(x)).$$

R Remember that composition is NOT commutative: $(f \circ g)(x) \neq (g \circ f)(x)$

■ **Example 1.4**
Let $f(x)\sin(x)$ and $g(x) = \frac{1}{x}$, we have:
- $(f \circ g)(x) = \sin\left(\frac{1}{x}\right)$ which is defined $\forall x \neq 0$,
- $(g \circ f)(x) = \frac{1}{\sin x}$ which is not defined $\forall x \in \mathbb{R}$ such that $\sin(x) = 0$ i.e $\forall x$ that are multiples of $\pm\pi$.

R In this course, we will assume that all basic properties, as well as their graphs, of special functions $(\sin, \cos, \tan, e, \ln, \log_a)$ are known.