2. Real numbers

2.1 The set of real numbers

It is clear that integers ($\mathbb{Z}$) represent only a certain category of numbers. Enlarging this set with all rational numbers is still inadequate to represent all possible numbers. Indeed, irrational numbers appear in calculus:

- area of a disc of radius 1: π,
- length of hypothenuse of a right isoceles triangle where each side has length 1: $\sqrt{2}$.

Proposition 2.1.1
There is no rational number x such that $x^2 = 2$.

Proof. We prove this proposition by contradiction. Assume that x is such a rational number. We assume, without loss of generality, that $x > 0$ and write $x = \frac{m}{n}$ where m and n are positive integers. Only three cases are possible:

1. m and n are of different types (i.e not both even or odd),
2. m and n are both odd,
3. m and n are both even.

First let notice a general property coming from the fact that $x = \sqrt{2}$. We have

$$x^2 = 2 \Leftrightarrow \frac{m^2}{n^2} = 2 \Leftrightarrow m^2 = 2n^2.$$

Thus m^2 is always an **even positive integer**.
Moreover, the third case is easy to address since if m and n were both even then it means that the ratio m/n is reducible to a fraction of either case 1) or 2). Therefore, we can assume that the third case returns us to the first two cases.
Let us now investigate cases 1) and 2).

1. Assume m and n are of different types. We have two different subcases:
   - assume that m is odd then $\exists k \in \mathbb{N}$ such that $m = 2k + 1$ hence

$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

This implies that m^2 is **odd** as well and we get a first contradiction since we showed above that m^2 is necessarily even.

- assume that m is even, then $\exists k \in \mathbb{N}$ such that $m = 2k$ hence

$$m^2 = 2n^2 \Leftrightarrow (2k)^2 = 2n^2 \Leftrightarrow 4k^2 = 2n^2 \Leftrightarrow 2k^2 = n^2.$$

This implies that n^2 is even and consequently that n is also **even** (otherwise, as for the m case above, if n was odd this will imply that n^2 is also odd). Thus We reach a second contradiction because we initially assumed that m and n are not of the same type.

2. Assume m and n are both odd. But from the previous case, we know that (independently of the type of n) m cannot be odd. Thus this case is not possible.

Therefore, we conclude that $\nexists m, n \in \mathbb{N}$ such that $x = \frac{m}{n}$, i.e x cannot be a rational number. ■

Definition 2.1.1
The set made of the union of the sets of rational numbers and the set of irrational numbers is defined as the set of real numbers and is denoted $\mathbb{R}$.

2.2 Arithmetic rules

$\forall x, y, z \in \mathbb{R}$, we have

- $x + y = y + x$ (addition is commutative),
- $x + (y + z) = (x + y) + z$ (addition is associative),
- $x + 0 = x$ (0 is the additive identity),
- $x + (-x) = 0$ ($-x$ is the additive inverse, we write $x - y$ for $x + (-y)$),
- $xy = yx$ (multiplication is commutative),
- $x(yz) = (xy)z$ (multiplication is associative),
- $1.x = x$ (1 is the multiplicative identity),
- $\forall x \neq 0, \exists \frac{1}{x}$ such that $x.\left(\frac{1}{x}\right) = 1$ ($\frac{1}{x}$ is the multiplicative inverse of x, we write $\frac{x}{y}$ for $x.\left(\frac{1}{y}\right)$),
- $x(y + z) = xy + xz$ (distributive law).

2.3 Order axioms

Real numbers are equipped with an **order relationship**:

- "$a < b$" means that a is less than b,
- "$a > b$" means that a is bigger than b.

The order relationship must follow the next rules:

1. $\forall a, b \in \mathbb{R}$, only one of the relationships

$$a = b \qquad ; \qquad a < b \qquad ; \qquad a > b$$

holds,

2. if $a < b$ then $\forall c \in \mathbb{R}$, we have $a + c < b + c$,
3. if $a > 0$ and $b > 0$ then $ab > 0$,
4. if $a > b$ and $b > c$ then $a > c$ (transitivity).

These rules lead to some useful inequality properties:

- if $a > b$ and $c > 0 \Rightarrow ac > bc$,
- if $a > b$ and $c < 0 \Rightarrow ac < bc$,
- if $0 < a < b \Rightarrow \frac{1}{a} > \frac{1}{b}$.

Notation 2.1.

$$a \le b \quad \textit{means} \quad a < b \quad \textit{or} \quad a = b,$$
$$a \ge b \quad \textit{means} \quad a > b \quad \textit{or} \quad a = b,$$

The next proposition is useful to prove some results.

Proposition 2.3.1
$\forall a, b \in \mathbb{R}$,

$$\forall \varepsilon > 0, a < b + \varepsilon \Rightarrow a \le b, \tag{2.1}$$
$$\forall \varepsilon > 0, a > b - \varepsilon \Rightarrow a \ge b. \tag{2.2}$$

Proof. It is sufficient to prove the first statement. Indeed, $\forall \varepsilon > 0$, if $a > b - \varepsilon$ then $\forall \varepsilon > 0, -a < -b + \varepsilon$. By the first statement, we have $-a \le -b \Leftrightarrow a \ge b$.
Let us prove the first statement. We will prove the contrapositive:

$$a > b \Rightarrow \exists \varepsilon > 0 \quad \text{such that} \quad a \ge b + \varepsilon.$$

Assume that $a > b$ then $\varepsilon = \frac{a-b}{2} > 0$, we have

$$b + \varepsilon = b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a$$

so that $a > b + \varepsilon$. ■

Corollary 2.3.2
$\forall \varepsilon > 0$, if $|a - b| < \varepsilon$ then $a = b$.

Proof. We have, $\forall \varepsilon > 0$,

$$|a - b| < \varepsilon \tag{2.3}$$
$$\Leftrightarrow -\varepsilon < a - b < \varepsilon \tag{2.4}$$
$$\Leftrightarrow b - \varepsilon < a < b + \varepsilon \tag{2.5}$$
$$\Leftrightarrow a \le b \quad \text{and} \quad a \ge b. \quad \text{(by the previous proposition)} \tag{2.6}$$

Therefore we have $a = b$. ■

2.4 The real line

The set $\mathbb{R}$ can be represented by a line. First, we need to chose an origin (corresponding to the number 0) and a unit length (corresponding to 1). Then each number x is at the position at a distance x from the origin, see Figure. 2.1. The sign of x indicates on which side of the origin we position x. Otherwise stating, the usual convention is

$$a < b \Leftrightarrow a \text{ is to the left of } b.$$

2.5 Intervals

We defined several types of intervals.

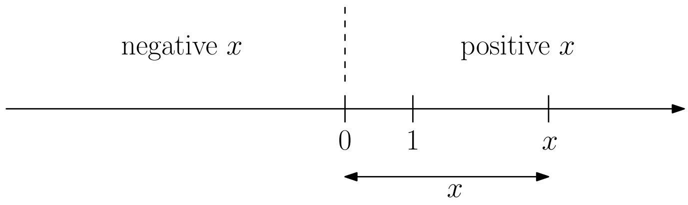

Figure 2.1: Real line. The origin is marked by 0, the unit length by 1.

Definition 2.5.1
Let $a, b \in \mathbb{R}$ with $a < b$ (a and b are called endpoints).
open interval: $(a,b) = \{x \in \mathbb{R} / a < x < b\}$ (does not contain its endpoints),
closed interval: $[a,b] = \{x \in \mathbb{R} / a \le x \le b\}$ (does contain its endpoints),
half-open intervals:

$$[a,b) = \{x \in \mathbb{R} / a \le x < b\},$$
$$(a,b] = \{x \in \mathbb{R} / a < x \le b\},$$

unbounded intervals:

$$(-\infty,b) = \{x \in \mathbb{R} / x < b\},$$
$$(-\infty,b] = \{x \in \mathbb{R} / x \le b\},$$
$$(a,+\infty) = \{x \in \mathbb{R} / x > a\},$$
$$[a,+\infty) = \{x \in \mathbb{R} / x \ge a\}.$$

2.6 Absolute value and triangle inequality

The absolute value measures the distance of a point to the origin.

Definition 2.6.1
$\forall x \in \mathbb{R}$, the absolute value of x, denoted $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

■ **Example 2.1**

$$|+5| = 5$$
$$|-5| = -(-5) = 5$$

■ **Example 2.2**
$\forall a, b \in \mathbb{R}$,

$$|a-b| = \begin{cases} a-b & \text{if } a \ge b \\ b-a & \text{if } a < b \end{cases}$$

The quantity $|a-b|$ represents the distance between a and b (e.g the distance between 2 and 7 is given by $|2-7| = 5$).

■ Example 2.3
Express

$$A = \{x \in \mathbb{R} / |x-3| \geq 7\}$$

as union of intervals.
Solution: $A =$ set of x whose distance from 3 is at least 7, i.e

$$A = (-\infty, -4] \cup [10, +\infty).$$

Proposition 2.6.1
$\forall a \in \mathbb{R}$ and $r \in \mathbb{R}, r > 0$, we have
- $\{x/|x-a| < r\} = (a-r, a+r)$ (open interval of length $2r$ centered at a),
- $\{x/|x-a| \leq r\} = [a-r, a+r]$ (closed interval of length $2r$ centered at a).

In particular, $\{x/|x| < r\} = (-r, r)$ and $\{x/|x| \leq r\} = [-r, r]$.

Proof. $\forall a \in \mathbb{R}$ and $r \in \mathbb{R}, r > 0$, pick a point x in $\{x/|x-a| < r\}$ then:

$$|x-a| < r \Leftrightarrow -r < x-a < r \quad (2.7)$$
$$\Leftrightarrow a-r < x < a+r \quad (2.8)$$
$$\Leftrightarrow x \in (a-r, a+r) \quad (2.9)$$

which proves the first statement. The second statement is obtained in the same way by changing $<$ by $\leq$. The particular cases are obtained by setting $a = 0$. ■

Proposition 2.6.2

$$\forall a, b \in \mathbb{R}, \quad |ab| = |a|.|b|$$

Proof. We need to consider four cases:
1. $a \geq 0$ and $b \geq 0$,
2. $a \geq 0$ and $b \leq 0$,
3. $a \leq 0$ and $b \geq 0$,
4. $a \leq 0$ and $b \leq 0$.

For each case, we have
1. $|a| = a$ and $|b| = b$ and $ab \geq 0$ thus $|ab| = ab = |a|.|b|$,
2. $|a| = a$ and $|b| = -b$ and $ab \leq 0$ thus $|ab| = -ab = a.(-b) = |a|.|b|$,
3. $|a| = -a$ and $|b| = b$ and $ab \leq 0$ thus $|ab| = -ab = (-a).b = |a|.|b|$,
4. $|a| = -a$ and $|b| = -b$ and $ab \geq 0$ thus $|ab| = ab = (-a).(-b) = |a|.|b|$,

■

Theorem 2.6.3 — Triangle inequality

$$\forall a, b \in \mathbb{R}, \qquad |a+b| \leq |a| + |b|.$$

Proof. Since

$$a = |a| \quad \text{or} \quad a = -|a|$$

and

$$b = |b| \quad \text{or} \quad b = -|b|,$$