Example 2.3
Express

$$A = \{x \in \mathbb{R} / |x-3| \geq 7\}$$

as union of intervals.
Solution: $A =$ set of x whose distance from 3 is at least 7, i.e

$$A = (-\infty, -4] \cup [10, +\infty).$$

Proposition 2.6.1
$\forall a \in \mathbb{R}$ and $r \in \mathbb{R}, r > 0$, we have
- $\{x/|x-a| < r\} = (a-r, a+r)$ (open interval of length $2r$ centered at a),
- $\{x/|x-a| \leq r\} = [a-r, a+r]$ (closed interval of length $2r$ centered at a).

In particular, $\{x/|x| < r\} = (-r, r)$ and $\{x/|x| \leq r\} = [-r, r]$.

Proof. $\forall a \in \mathbb{R}$ and $r \in \mathbb{R}, r > 0$, pick a point x in $\{x/|x-a| < r\}$ then:

$$\begin{aligned} |x-a| < r &\Leftrightarrow -r < x-a < r && (2.7)\\ &\Leftrightarrow a-r < x < a+r && (2.8)\\ &\Leftrightarrow x \in (a-r, a+r) && (2.9) \end{aligned}$$

which proves the first statement. The second statement is obtained in the same way by changing $<$ by $\leq$. The particular cases are obtained by setting $a = 0$. ■

Proposition 2.6.2

$$\forall a, b \in \mathbb{R}, \quad |ab| = |a|.|b|$$

Proof. We need to consider four cases:
1. $a \geq 0$ and $b \geq 0$,
2. $a \geq 0$ and $b \leq 0$,
3. $a \leq 0$ and $b \geq 0$,
4. $a \leq 0$ and $b \leq 0$.

For each case, we have
1. $|a| = a$ and $|b| = b$ and $ab \geq 0$ thus $|ab| = ab = |a|.|b|$,
2. $|a| = a$ and $|b| = -b$ and $ab \leq 0$ thus $|ab| = -ab = a.(-b) = |a|.|b|$,
3. $|a| = -a$ and $|b| = b$ and $ab \leq 0$ thus $|ab| = -ab = (-a).b = |a|.|b|$,
4. $|a| = -a$ and $|b| = -b$ and $ab \geq 0$ thus $|ab| = ab = (-a).(-b) = |a|.|b|$,

■

Theorem 2.6.3 — Triangle inequality

$$\forall a, b \in \mathbb{R}, \qquad |a+b| \leq |a| + |b|.$$

Proof. Since

$$a = |a| \quad \text{or} \quad a = -|a|$$

and

$$b = |b| \quad \text{or} \quad b = -|b|,$$

we have

$$-|a| \leq a \leq |a| \qquad \text{and} \qquad -|b| \leq b \leq |b|.$$

Therefore,

$$-|a| - |b| \leq a + b \leq |a| + |b|$$
$$\Leftrightarrow -(|a| + |b|) \leq a + b \leq |a| + |b|.$$

Let $c = |a| + |b|$ and $d = a + b$, we have

$$-c \leq d \leq c$$
$$\Leftrightarrow |d| \leq c$$
$$\Leftrightarrow |a+b| \leq |a| + |b|.$$

■

Corollary 2.6.4

$$\forall a, b \in \mathbb{R}, \qquad \big||a| - |b|\big| \leq |a - b|.$$

Proof. By the triangle inequality, we have

$$|a| = |a - b + b| \leq |a - b| + |b|$$

so that

$$|a| - |b| \leq |a - b|.$$

Similarly,

$$|b| = |b - a + a| \leq |b - a| + |a| = |a - b| + |a|$$

so that

$$|b| - |a| \leq |a - b| \Leftrightarrow |a| - |b| \geq -|a - b|.$$

Thus

$$-|a - b| \leq |a| - |b| \leq |a - b| \Leftrightarrow \big||a| - |b|\big| \leq |a - b|.$$

■

2.7 The Archimedean property of $\mathbb{R}$

We will assume known the principle of mathematical induction.

Theorem 2.7.1
Let S be a subset of $\mathbb{N}$. Assume that $N \in S$ and $n + 1 \in S$ if $n \in S$. Then

$$S = \{n \in \mathbb{N} / n \geq N\}.$$

In particular, if $1 \in S$ and $n \in S$ implies that $n + 1 \in S$ then $S = \mathbb{N}$.

Definition 2.7.1
A subset R of $\mathbb{R}$ is said to be **well-ordered** if each nonempty subset of R has a smallest element.

Proposition 2.7.2
The set $\mathbb{N}$ is well-ordered.

Proof. We use a proof by contradiction:
Assume S is a nonempty subset of $\mathbb{N}$ that does NOT have a smallest element. Let T denote the complement of S in $\mathbb{N}$. We will show that $T = \mathbb{N}$ which contradicts the fact that S is nonempty.
If $1 \in S$ then 1 is the smallest element. Therefore, $1 \notin S$ and $1 \in T$.
Let T' be the subset of T consisting of all $n \in T$ such that $1, 2, 3, \ldots, n \in T'$. We need to show that $n+1 \in T'$ as well.
Assume this is not the case, thus $n+1 \in S$. Since $1, 2, 3, \ldots, n \in T$, which is the complement of S, the number $n+1$ must be the smallest element of S. But we assumed that S does not have a smallest element. Thus $n+1 \in T'$. Therefore $T' = \mathbb{N}$ implying $T = \mathbb{N}$, thus S is empty and we get our contradiction and conclude that S must have a smallest element. ■

Definition 2.7.2
Let S be a subset of $\mathbb{R}$. We say that S has the **Archimedean property** if $\forall x \in S, \exists n \in \mathbb{N}$ such that $x < n$.

■ **Example 2.4**
$\mathbb{Q}$ has the Archimedean property. Indeed, let $x \in \mathbb{Q}$ and $x \leq 1$ then we can set $n = 2$. If $x > 0$, we can write $x = p/q$ where $p, q \in \mathbb{N}$. Thus $p \geq 1, q \geq 1 \Leftrightarrow \frac{1}{q} \leq 1 \Leftrightarrow \frac{p}{q} \leq p$. Therefore, $x = \frac{p}{q} \leq p < p+1$.

Proposition 2.7.3
$\mathbb{R}$ has the Archimedean property.

The proof of this proposition is out of the scope of this course.

Proposition 2.7.4
$\forall x, y \in \mathbb{R}, x > 0, y > 0, \exists n \in \mathbb{N}$ such that $nx > y$.
In particular, $\forall x \in \mathbb{R}, x > 0, \exists m \in \mathbb{N}$ such that $\frac{1}{m} < x$.

Proof. Let $x, y \in \mathbb{R}, x > 0, y > 0$ and denote $z = \frac{y}{x}$. By the Archimedean property of $\mathbb{R}, \exists n \in \mathbb{N}$ such that $n > z = \frac{y}{x}$. Thus $nx > y$.
In particular, $\forall x > 0, \exists m \in \mathbb{N}$ such that $mx > 1$ thus $x > \frac{1}{m}$. ■