3. Sequences and limits

3.1 Sequences

Definition 3.1.1
A sequence is a function whose domain is a subset of integers of the form $\{N_0, N_0+1, N_0+2, \dots\}$ where $N_0 \in \mathbb{N}$. If we refer to this function as f, then $f(n)$ is usually denoted a_n for $n = N_0, N_0+1, N_0+2, \dots$ The term a_n is called the $n-$th term of the sequence.

R We may write $a_{N_0}, a_{N_0+1}, a_{N_0+2}, \dots$ or $\{a_n\}_{n=N_0}^{\infty}$ or simply $\{a_n\}$ if the starting value N_0 is straightforward.

Example 3.1
The sequence

$$1; \frac{1}{2}; \frac{1}{3}; \frac{1}{4}; \dots; \frac{1}{n}; \dots$$

can be denoted $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ or $\left\{\frac{1}{n}\right\}$.

Example 3.2
The sequence

$$5; \frac{6}{2}; \frac{7}{3}; \frac{8}{4}; \dots$$

can be denoted $\left\{\frac{n}{n-4}\right\}_{n=5}^{\infty}$.

R The index n is a "dummy index" and can be replaced by any other letter, i.e $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{k}\right\}_{k=1}^{\infty}$.

Definition 3.1.2
The graph of a sequence $\{a_n\}_{n=N_0}^{\infty}$ is the set of points of coordinates (n, a_n) in the Cartesian plane, where $n = N_0, N_0+1, N_0+2, \ldots$
The range of the sequence $\{a_n\}_{n=N_0}^{\infty}$ is the range of the function f such that $f(n) = a_n, n \geq N_0$.

Example 3.3
Let $a_n = \frac{n}{n-4}$ for $n = 5, 6, 7, \ldots$ The graph of $\{a_n\}_{n=5}^{\infty}$ is the set of points $\{(5,5), (6,3), (7,7/3), \ldots\}$ and is represented in the Cartesian plane on the left of Figure. 3.1 while the right plot shows the representation along the real line.

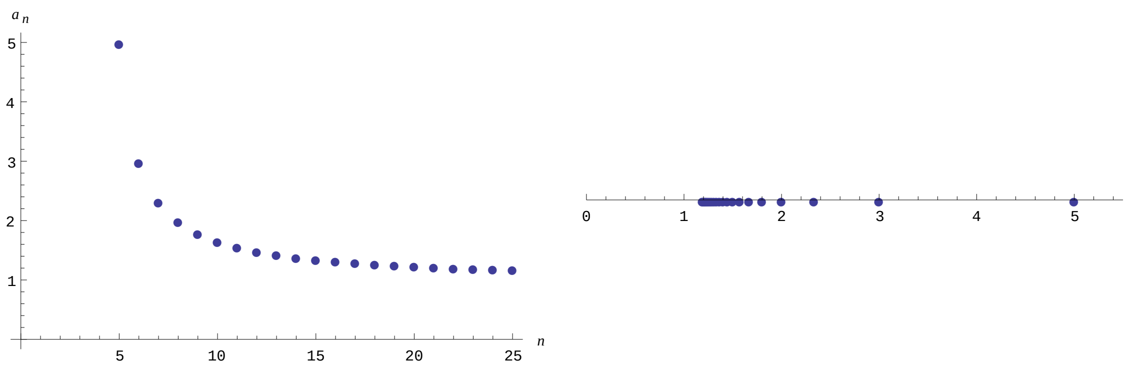

Figure 3.1: Graphs of the sequence $a_n = \frac{n}{n-4}$ for $n = 5; 6; \ldots; 25$ used in Example 3.3. The left graph is the Cartesian graph. The right graph is the plot along the real line.

3.2 Limits

Intuitively, the notion of limits can be defined in the following way: L is said to be the limit of $\{a_n\}$ if a_n is as close to L as desired, provided that n is sufficiently large. A precise definition of a limit is given by

Definition 3.2.1
The limit of the sequence $\{a_n\}$ is L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |a_n - L| < \varepsilon,$$

we write

$$\lim_{n \to \infty} a_n = L.$$

R Note that in the general case, N depends on the choice of ε. It is sometimes denoted $N(\varepsilon)$ or N_ε.

Geometric interpretation: the quantity $|a_n - L| < \varepsilon$ is equivalent to $L - \varepsilon < a_n < L + \varepsilon$. Therefore, the previous definition corresponds to the fact that for n large enough ($n \geq N$) all values a_n belong to the interval $(L - \varepsilon, L + \varepsilon)$ (see Figure 3.2). Because ε is an arbitrary number, in particular we can choose it as small as we want, i.e we can shrink the interval as we want up to get the singleton $\{L\}$.

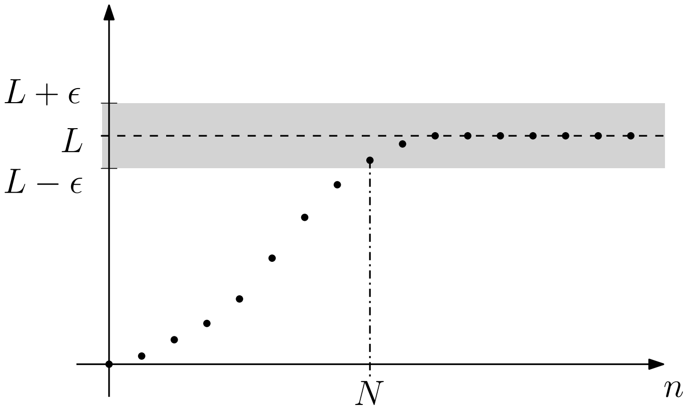

Figure 3.2: Geometric interpretation of the limit of a sequence.

■ **Example 3.4**
Show that the limit of $\left\{\frac{n}{n-4}\right\}_{n=5}^{\infty}$ is $L=1$.
Solution: we want to show that $\forall \varepsilon > 0, \exists N \in \mathbb{N}, N \geq 5, \forall n \in \mathbb{N}, n \geq N, |a_n - 1| < \varepsilon$.
We have

$$|a_n - 1| = \left|\frac{n}{n-4} - 1\right| = \left|\frac{n-n+4}{n-4}\right| = \frac{4}{n-4} \underset{(n\geq N)}{\leq} \frac{4}{N-4}.$$

Given $\varepsilon > 0$, if we choose N such that $\frac{4}{N-4} < \varepsilon$ then we automatically have $|a_n - 1| < \varepsilon$. But, we have

$$\frac{4}{N-4} < \varepsilon \Leftrightarrow \frac{4}{\varepsilon} < N-4 \Leftrightarrow N > \frac{4}{\varepsilon} + 4$$

Therefore any N such that $N > \frac{4}{\varepsilon} + 4$ works (and it exists because $\mathbb{R}$ is Archimedean). Finally, we can write

$$\lim_{n\to\infty} \frac{n}{n-4} = 1.$$

Proposition 3.2.1
The limit of a sequence is unique.

Proof. Assume $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} a_n = L_2$. We want to prove that $L_1 = L_2$ by showing that $\forall \varepsilon > 0, |L_1 - L_2| < \varepsilon$.
Using the definition of a limit, we have, $\forall \varepsilon > 0$,

$$\exists N_1 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N_1, |a_n - L_1| < \frac{\varepsilon}{2}$$

and

$$\exists N_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N_2, |a_n - L_2| < \frac{\varepsilon}{2}.$$

Therefore, if we set $N = \max(N_1, N_2)$, we have

$$|a_N - L_1| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |a_N - L_2| < \frac{\varepsilon}{2}.$$

Thus

$$|L_1 - L_2| = |(L_1 - a_N) + (a_N - L_2)| \underset{\text{triangle inequality}}{\leq} |L_1 - a_N| + |L_2 - a_N| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

Definition 3.2.2
A subsequence of a sequence $\{a_n\}_{n=1}^{\infty}$ is formed by selecting the terms a_n that correspond to the values of n taken as a strictly increasing sequence: if

$$n_1 < n_2 < n_3 < \ldots < n_k < n_{k+1} < \ldots$$

is a strictly increasing sequence of integers, the corresponding sequence $\{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Example 3.5
Let

$$\{a_n\}_{n=1}^{\infty} = \left\{(-1)^{n+1}\frac{1}{n}\right\}_{n=1}^{\infty} = 1; -\frac{1}{2}; \frac{1}{3}; -\frac{1}{4}; \frac{1}{5}; -\frac{1}{6}; \ldots.$$

The subsequence of $\{a_n\}$ corresponding to odd values of n is obtained by setting

$$\{n_k\}_{k=1}^{\infty} = \{2k-1\}_{k=1}^{\infty} = 1; 3; 5; 7; \ldots$$

The corresponding subsequence is

$$\{a_{n_k}\}_{k=1}^{\infty} = \left\{(-1)^{n_k+1}\frac{1}{n_k}\right\}_{k=1}^{\infty} = \left\{(-1)^{2k-1+1}\frac{1}{2k-1}\right\}_{k=1}^{\infty} = \left\{\frac{1}{2k-1}\right\}_{k=1}^{\infty} = 1; \frac{1}{3}; \frac{1}{5}; \frac{1}{7}; \ldots$$

To get the subsequence corresponding to even values of n, we set

$$\{n_k\}_{k=1}^{\infty} = \{2k\}_{k=1}^{\infty} = 2; 4; 6; \ldots$$

and the corresponding subsequence is

$$\{a_{n_k}\}_{k=1}^{\infty} = \left\{(-1)^{n_k+1}\frac{1}{n_k}\right\}_{k=1}^{\infty} = \left\{(-1)^{2k+1}\frac{1}{2k}\right\}_{k=1}^{\infty} = \left\{-\frac{1}{2k}\right\}_{k=1}^{\infty} = -\frac{1}{2}; -\frac{1}{4}; -\frac{1}{6}; \ldots$$

Proposition 3.2.2
Let $\{a_n\}$ be a sequence converging to a limit L. Then all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ converge to L as well.

Proof. Let $\{a_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{a_n\}_{n=1}^{\infty}$. Since we assume that $\lim_{n\to\infty} a_n = L$, we have

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - L| < \varepsilon.$$

Thus there exists $K \in \mathbb{N}$ such that $\forall k \geq K, n_k \geq N$ (because the set $n_1, n_2, \ldots$ is increasing) such that $|a_{n_k} - L| < \varepsilon$. Finally, this statement reduces to

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \geq K, |a_{n_k} - L| < \varepsilon$$

which is equivalent to write

$$\lim_{k\to\infty} a_{n_k} = L.$$

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