5. Continuity of functions

5.1 Formal definitions

The notion of continuity of a function can be easily defined from the notion of limits.

Definition 5.1.1
A function $f : D \to R \subset \mathbb{R}$ is said to be **continuous at** $x_0 \in D$ if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

or equivalently

$$\forall \varepsilon > 0, \exists \delta > 0, \forall h \in D, |h| < \delta \Rightarrow |f(x_0 + h) - f(x_0)| < \varepsilon.$$

Therefore, every technique used to find limits can be used to prove continuity.

Definition 5.1.2
A function $f : D \to R \subset \mathbb{R}$ is said to be continuous on its domain if it is continuous $\forall x_0 \in D$.

■ Example 5.1
Show that $f(x) = x^2$ is continuous over $\mathbb{R}$.
Solution: we need to show

$$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

We have

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |(x + x_0)(x - x_0)| = |x + x_0||x - x_0| \leq (|x| + |x_0|)|x - x_0|.$$

Given the fact that we want $x \to x_0$, we can consider $x_0 - 1 < x < x_0 + 1 \Leftrightarrow |x - x_0| < 1$. Thus,

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < 1 + |x_0|$$

$$\Rightarrow |x| + |x_0| < 1 + 2|x_0|.$$

Therefore

$$|f(x) - f(x_0)| \leq (|x| + |x_0|)|x - x_0| < (1 + 2|x_0|)|x - x_0|.$$

In order to get $\forall \varepsilon > 0, |f(x) - f(x_0)| < \varepsilon$, it is sufficient to choose $(1 + 2|x_0|)|x - x_0| < \varepsilon \Leftrightarrow |x - x_0| < \frac{\varepsilon}{1+2|x_0|}$. Remember that we also need to have $|x - x_0| < 1$, we can set $\delta = \min\left(1, \frac{\varepsilon}{1+2|x_0|}\right)$ and we get

$$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \delta = \min\left(1, \frac{\varepsilon}{1+2|x_0|}\right), \forall x \in \mathbb{R}, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

$$\Leftrightarrow \forall x_0 \in \mathbb{R}, \lim_{x \to x_0} f(x) = f(x_0).$$

i.e $f(x) = x^2$ is continuous $\forall x_0 \in \mathbb{R}$.

Definition 5.1.3 — Left and Right continuity.

- A function $f : D \to R \subset \mathbb{R}$ is said to be **continuous on the left at** $x_0 \in D$ if

$$\lim_{x \to x_0-} f(x) = f(x_0).$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x_0 - \delta < x < x_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

- A function $f : D \to R \subset \mathbb{R}$ is said to be **continuous on the right at** $x_0 \in D$ if

$$\lim_{x \to x_0+} f(x) = f(x_0).$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x_0 < x < x_0 + \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

R Note that f is continuous at x_0 if f is continuous on the right and the left at x_0, i.e

$$\lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow \left(\lim_{x \to x_0+} f(x) = f(x_0)\right) \wedge \left(\lim_{x \to x_0-} f(x) = f(x_0)\right).$$

■ **Example 5.2**

Let $\forall x \geq 0, f(x) = \sqrt{x}$. Show that f is continuous on $[0, +\infty)$.

Solution: We need to consider two cases: the continuity of f when $x_0 \in (0, +\infty)$ and at the endpoint 0. Let us start with the first case: we want to prove that $\forall x_0 \in (0, +\infty), \lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$, i.e

$$\forall x_0 \in (0, +\infty), \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (0, +\infty), |x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon.$$

But
$$|\sqrt{x}-\sqrt{x_0}| = \left|(\sqrt{x}-\sqrt{x_0})\frac{\sqrt{x}+\sqrt{x_0}}{\sqrt{x}+\sqrt{x_0}}\right| = \left|\frac{x-x_0}{\sqrt{x}+\sqrt{x_0}}\right|.$$
We also have
$$\sqrt{x}+\sqrt{x_0} > \sqrt{x_0} \Leftrightarrow \frac{1}{\sqrt{x}+\sqrt{x_0}} < \frac{1}{\sqrt{x_0}}.$$
Thus
$$|\sqrt{x}-\sqrt{x_0}| = \left|\frac{x-x_0}{\sqrt{x}+\sqrt{x_0}}\right| < \frac{|x-x_0|}{\sqrt{x_0}}.$$
In order to get $\forall \varepsilon > 0, |\sqrt{x}-\sqrt{x_0}| < \varepsilon$, we can choose $\frac{|x-x_0|}{\sqrt{x_0}} < \varepsilon \Leftrightarrow |x-x_0| < \varepsilon\sqrt{x_0}$. Therefore,
$$\forall x_0 \in (0,+\infty), \forall \varepsilon > 0, \exists \delta > 0, \forall x \in (0,+\infty), |x-x_0| < \delta \Rightarrow |\sqrt{x}-\sqrt{x_0}| < \varepsilon$$
$$\Leftrightarrow \forall x_0 \in (0,+\infty), \lim_{x\to x_0}\sqrt{x} = \sqrt{x_0}.$$
and we get that f is continuous at all $x_0 \in (0,+\infty)$.
We now check the continuity at the endpoint 0: we want to prove that $\lim_{x\to 0+}\sqrt{x} = \sqrt{0} = 0$, i.e
$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (0,+\infty), 0 < x < 0+\delta \Rightarrow |\sqrt{x}-0| < \varepsilon.$$
Since we are interested when $x \to 0+$, we have $x > 0$. On the other hand, we have $|\sqrt{x}-0| = \sqrt{x}$ thus in order to have $\forall \varepsilon > 0, |\sqrt{x}-0| < \varepsilon$ we can choose $\sqrt{x} < \varepsilon \Leftrightarrow x < 0+\varepsilon^2$. Therefore, if we set $\delta = \varepsilon^2$ we have
$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (0,+\infty), 0 < x < 0+\delta \Rightarrow |\sqrt{x}-0| < \varepsilon \Leftrightarrow \lim_{x\to 0+} f(x) = f(0).$$
and we get that f is continuous on the right at 0.
Finally, we conclude that f is continuous on $[0,+\infty)$.

5.2 Continuity and convergence of sequences

We can also characterize continuity by the limit of sequences:

Proposition 5.2.1 — Sequential characterization of continuity.
Let a function $f : D \to \mathbb{R}$ (D being an open interval) and a point $x_0 \in D$. Then f is continuous at $x_0 \Leftrightarrow$ for any sequence $\{x_n\}$ where $\forall n \in \mathbb{N}, x_n \in D$ and such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Proof. It is a direct consequence of the sequential characterization of limits. ■

R This theorem is equivalent to state that if f is continuous at x_0 then for any sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$, we have
$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right).$$

This theorem is useful to prove that a function is not continuous at a point x_0. Indeed, its contraposition states that if
$$\exists\{x_n\}, (\lim_{n\to\infty} x_n = x_0) \wedge (\lim_{n\to\infty} f(x_n) \text{ does not exists})$$
then f is discontinuous at x_0.

■ **Example 5.3**
Let

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$

Show that f is discontinuous at 0.
Solution: we will find a sequence $\{x_n\}$ which converges to 0 but where $\{f(x_n)\}$ does not converge, i.e there exists two subsequences $\{x_{1,n}\}$ and $\{x_{2,n}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} f(x_{1,n}) \neq \lim_{n\to\infty} f(x_{2,n})$.
Consider the sequence

$$\forall n \in \mathbb{N}, x_n = (-1)^n \frac{1}{n}.$$

Clearly, $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} f(x_n)$ does not converge. Indeed, if we consider the two subsequences for odd and even n, we have:

- if n is even then $\forall k \in \mathbb{N}, n = 2k$ and $x_{2k} = (-1)^{2k}\frac{1}{2k} = \frac{1}{2k} > 0$ thus $\forall k \in \mathbb{N}, f(x_{2k}) = 1 \Rightarrow \lim_{k\to\infty} f(x_{2k}) = 1$.
- if n is odd then $\forall k \in \mathbb{N}, n = 2k-1$ and $x_{2k-1} = (-1)^{2k-1}\frac{1}{2k-1} = -\frac{1}{2k} < 0$ thus $\forall k \in \mathbb{N}, f(x_{2k-1}) = -1 \Rightarrow \lim_{k\to\infty} f(x_{2k-1}) = -1$.

Therefore, f is discontinuous at 0.

5.3 Uniform continuity

When we use the formal definition of continuity, the variable δ generally depends on x_0. Uniform continuity is the special case when we can find a value for δ which will work for all x_0, i.e δ does not depend on x_0. This leads to the following definition.

Definition 5.3.1
A function f is said to be **uniformly continuous on an interval** $D \subset \mathbb{R}$ if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in D, |x_2 - x_1| < \delta \Rightarrow |f(x_2) - f(x_1)| < \varepsilon.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, \forall h, x+h \in D, |h| < \delta \Rightarrow |f(x+h) - f(x)| < \varepsilon.$$

■ **Example 5.4**
Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $[1/2, +\infty)$.
Solution: We need to prove that

$$\forall \varepsilon > 0, \exists \delta, \forall x_1 \geq \frac{1}{2}, \forall x_2 \geq \frac{1}{2}, |x_1 - x_2| < \delta \Rightarrow |f(x_2) - f(x_1)| < \varepsilon.$$

Assume that $x_1 \geq 1/2$ and $x_2 \geq 1/2$. We have

$$|f(x_2) - f(x_1)| = \left|\frac{1}{x_2} - \frac{1}{x_1}\right| = \left|\frac{x_1 - x_2}{x_1 x_2}\right| = \left|\frac{x_2 - x_1}{x_1 x_2}\right|.$$

Since $x_1 \geq 1/2$ and $x_2 \geq 1/2$, we have $x_1 x_2 \geq 1/4 \Leftrightarrow \frac{1}{x_1 x_2} \leq 4$. Thus

$$|f(x_2) - f(x_1)| = \left|\frac{x_2 - x_1}{x_1 x_2}\right| \leq 4|x_2 - x_1|.$$