■ **Example 5.3**
Let
$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}.$$
Show that f is discontinuous at 0.
Solution: we will find a sequence $\{x_n\}$ which converges to 0 but where $\{f(x_n)\}$ does not converge, i.e there exists two subsequences $\{x_{1,n}\}$ and $\{x_{2,n}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} f(x_{1,n}) \neq \lim_{n\to\infty} f(x_{2,n})$.
Consider the sequence
$$\forall n \in \mathbb{N}, x_n = (-1)^n \frac{1}{n}.$$
Clearly, $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} f(x_n)$ does not converge. Indeed, if we consider the two subsequences for odd and even n, we have:
- if n is even then $\forall k \in \mathbb{N}, n = 2k$ and $x_{2k} = (-1)^{2k}\frac{1}{2k} = \frac{1}{2k} > 0$ thus $\forall k \in \mathbb{N}, f(x_{2k}) = 1 \Rightarrow \lim_{k\to\infty} f(x_{2k}) = 1$.
- if n is odd then $\forall k \in \mathbb{N}, n = 2k-1$ and $x_{2k-1} = (-1)^{2k-1}\frac{1}{2k-1} = -\frac{1}{2k} < 0$ thus $\forall k \in \mathbb{N}, f(x_{2k-1}) = -1 \Rightarrow \lim_{k\to\infty} f(x_{2k-1}) = -1$.

Therefore, f is discontinuous at 0.

5.3 Uniform continuity

When we use the formal definition of continuity, the variable δ generally depends on x_0. Uniform continuity is the special case when we can find a value for δ which will work for all x_0, i.e δ does not depend on x_0. This leads to the following definition.

Definition 5.3.1
A function f is said to be **uniformly continuous on an interval** $D \subset \mathbb{R}$ if and only if
$$\forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in D, |x_2 - x_1| < \delta \Rightarrow |f(x_2) - f(x_1)| < \varepsilon.$$
$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, \forall h, x+h \in D, |h| < \delta \Rightarrow |f(x+h) - f(x)| < \varepsilon.$$

■ **Example 5.4**
Let $f(x) = \frac{1}{x}$. Show that f is uniformly continuous on $[1/2, +\infty)$.
Solution: We need to prove that
$$\forall \varepsilon > 0, \exists \delta, \forall x_1 \geq \frac{1}{2}, \forall x_2 \geq \frac{1}{2}, |x_1 - x_2| < \delta \Rightarrow |f(x_2) - f(x_1)| < \varepsilon.$$
Assume that $x_1 \geq 1/2$ and $x_2 \geq 1/2$. We have
$$|f(x_2) - f(x_1)| = \left|\frac{1}{x_2} - \frac{1}{x_1}\right| = \left|\frac{x_1 - x_2}{x_1 x_2}\right| = \left|\frac{x_2 - x_1}{x_1 x_2}\right|.$$
Since $x_1 \geq 1/2$ and $x_2 \geq 1/2$, we have $x_1 x_2 \geq 1/4 \Leftrightarrow \frac{1}{x_1 x_2} \leq 4$. Thus
$$|f(x_2) - f(x_1)| = \left|\frac{x_2 - x_1}{x_1 x_2}\right| \leq 4|x_2 - x_1|.$$

In order to have $\forall\varepsilon > 0, |f(x_2) - f(x_1)| < \varepsilon$, we can choose $4|x_2 - x_1| < \varepsilon \Leftrightarrow |x_2 - x_1| < \frac{\varepsilon}{4}$. Therefore, if we set $\delta = \frac{\varepsilon}{4}$ we get

$$\forall\varepsilon > 0, \exists\delta, \delta = \frac{\varepsilon}{4}, \forall x_1 \geq \frac{1}{2}, \forall x_2 \geq \frac{1}{2}, |x_1 - x_2| < \delta \Rightarrow |f(x_2) - f(x_1)| < \varepsilon$$

$\Leftrightarrow f$ is uniformly continuous on $[1/2, +\infty)$.

Theorem 5.3.1 — Sequential characterization of uniform continuity
A function $f : D \to \mathbb{R}$ is uniformly continuous on D if and only if

$$\forall\{u_n\} \in D, \forall\{v_n\} \in D, \lim_{n\to\infty}(u_n - v_n) = 0 \Rightarrow \lim_{n\to\infty}(f(u_n) - f(v_n)) = 0.$$

Proof. $\Rightarrow$: Assume f is uniformly continuous and let $\{u_n\}$ and $\{v_n\}$ two sequences of elements of D such that $\lim_{n\to\infty}(u_n - v_n) = 0$. We need to show that $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$.
Since f is uniformly continuous, we have

$$\forall\varepsilon > 0, \exists\delta > 0, \forall x_1, x_2 \in D, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

Since $\lim_{n\to\infty}(u_n - v_n) = 0$, we have

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |u_n - v_n| < \delta.$$

Therefore,

$$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |u_n - v_n| < \delta \Rightarrow |f(u_n) - f(v_n)| < \varepsilon$$
$$\Leftrightarrow \lim_{n\to\infty}(f(u_n) - f(v_n)) = 0.$$

$\Leftarrow$: Assume that

$$\forall\{u_n\} \in D, \forall\{v_n\} \in D, \lim_{n\to\infty}(u_n - v_n) = 0 \Rightarrow \lim_{n\to\infty}(f(u_n) - f(v_n)) = 0.$$

We proceed by contradiction: assume that f is not uniformly continuous, i.e

$$\exists\varepsilon > 0, \forall\delta > 0, \exists x_1, x_2 \in D, (|x_2 - x_1| < \delta) \wedge (|f(x_2) - f(x_1)| \geq \varepsilon).$$

Choose two sequences $\{u_n\}$ and $\{v_n\}$ such that $\lim_{n\to\infty}(u_n - v_n) = 0$ thus

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |u_n - v_n| < \delta \wedge |f(u_n) - f(v_n)| \geq \varepsilon.$$
$$\Leftrightarrow \lim_{n\to\infty}(f(u_n) - f(v_n)) \neq 0.$$

This contradict the initial assumption. Therefore f must be uniformly continuous. ■

As with the standard continuity, the sequential characterization of uniform continuity is useful in practice to prove that a function is not uniformly continuous. By the contraposition of the previous theorem, it is sufficient to find two sequences such $\lim_{n\to\infty}(u_n - v_n) = 0$ and $\lim_{n\to\infty}(f(u_n) - f(v_n)) \neq 0$ to prove that f is not uniformly continuous.

■ Example 5.5
Let $\forall x \neq 0, f(x) = \frac{1}{x}$. Show that f is not uniformly continuous on $(0, 1]$.
Solution: Choose $u_n = \frac{1}{n}$ and $v_n = \frac{1}{n+1}$. Then

$$\lim_{n\to\infty}(u_n - v_n) = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0.$$

But $f(u_n) - f(v_n) = n - (n+1) = -1$ thus $\lim_{n\to\infty}(f(u_n) - f(v_n)) = -1 \neq 0$. Therefore f is not uniformly continuous on $(0,1]$.

5.3.1 Properties of continuity

In this subsection, we study the continuity of basic functions and the algebra of continuity. Here, we only give the result statements, their corresponding proofs will be part of the next homework assignment.

Proposition 5.3.2
A constant function is continuous over $\mathbb{R}$.

Proof. Let $\forall x \in \mathbb{R}, f(x) = c$ where $c \in \mathbb{R}$ is constant. We want to prove that

$$\forall x_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \forall x \in \mathbb{R}, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

But we have $f(x) - f(x_0)| = |c - c| = 0$ thus we can arbitrarily choose $\delta > 0$ and will always have $\forall \varepsilon > 0, |f(x) - f(x_0)| < \varepsilon$. ■

Proposition 5.3.3
Let the function $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, f_n(x) = x^n$. Then f_n is continuous over $\mathbb{R}$.

Proof. See homework. ■

Theorem 5.3.4
Let f and g be two functions continuous at x_0 then

- $f + g$ is continuous at x_0,
- fg is continuous at x_0,
- f/g is continuous at x_0 providing that $g(x_0) \neq 0$.

Proof. See homework. ■

Proposition 5.3.5
A polynomial is continuous over $\mathbb{R}$.

Proof. See homework. ■

Proposition 5.3.6
A rational function is continuous on its domain of definition.

Proof. See homework. ■

Proposition 5.3.7
Trigonometric functions (those defined over $\mathbb{R}$) and the exponential are continuous over $\mathbb{R}$, the logarithm function is continuous over $(0, +\infty)$.

Proof. The corresponding proofs need some tools which will be studied later. ■

Theorem 5.3.8
Assume that f is continuous at x_0 and g is continuous at $f(x_0)$. Then the composite function $g \circ f$ is continuous at x_0.

Proof. Since g is continuous at $f(x_0)$, we have

$$\forall \varepsilon > 0, \exists \delta_1 > 0, \forall x, |f(x) - f(x_0)| < \delta_1 \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon.$$

On the other, since f is continuous at x_0, we have

$$\forall \delta_1 > 0, \exists \delta > 0, \forall x, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \delta_1.$$

Therefore,

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, |x - x_0| < \delta \Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon$$

$\Leftrightarrow g \circ f$ is continuous at x_0. ■

Corollary 5.3.9
Let $f : D \to \mathbb{R}$ and let x_0 (eventually not contained in D) such that $\lim_{x \to x_0} f(x) = L$. Assume that g is continuous at L. Then

$$\lim_{x \to x_0} g(f(x)) = g(L) = g\left(\lim_{x \to x_0} f(x)\right).$$

Proof. Left as an exercise. ■

■ **Example 5.6**
Find

$$\lim_{x \to 2} \sin\left(\frac{\pi(x^2 - 4)}{(x - 2)}\right).$$

Solution: Since the sin function is continuous over $\mathbb{R}$, we have

$$\lim_{x \to 2} \sin\left(\frac{\pi(x^2 - 4)}{(x - 2)}\right) = \sin\left(\lim_{x \to 2} \frac{\pi(x^2 - 4)}{(x - 2)}\right).$$

But

$$\lim_{x \to 2} \frac{\pi(x^2 - 4)}{(x - 2)} = \lim_{x \to 2} \frac{\pi(x - 2)(x + 2)}{(x - 2)} = \lim_{x \to 2} \pi(x + 2) = 4\pi.$$

Therefore

$$\lim_{x \to 2} \sin\left(\frac{\pi(x^2 - 4)}{(x - 2)}\right) = \sin(4\pi) = 0.$$

5.4 Consequences of continuity

5.4.1 Extreme value theorem

Theorem 5.4.1 — Extreme value theorem
Assume that f is continuous on a closed and bounded interval $[a, b]$. Then f is bounded and attains its (absolute) maximum and minimum values on $[a, b]$.

Proof. First let us show that f is bounded. We proceed by contradiction: assume that f is not bounded then

$$\forall n \in \mathbb{N}, \exists x_n \in [a,b], |f(x_n)| \geq n.$$

If we consider the sequence $\{x_n\}$, it is clear that this sequence is bounded (since all its elements are in a bounded interval). Then from the Bolzano-Weierstrass theorem, we know that there exists a converging subsequence $\{x_{n_k}\}$ such that

$$\lim_{k \to \infty} x_{n_k} = x_0 \in [a,b].$$

Since f is continuous, from the sequential characterization of continuity, we have that

$$\lim_{n \to \infty} f(x_n) = f(x_0).$$

This is in contradiction with the assumption that the sequence $\{f(x_{n_k})\}$ does not converge. Thus f must be bounded.
Now, we need to prove that f attains its maximum and minimum. Since we proved that f is bounded, the quantity

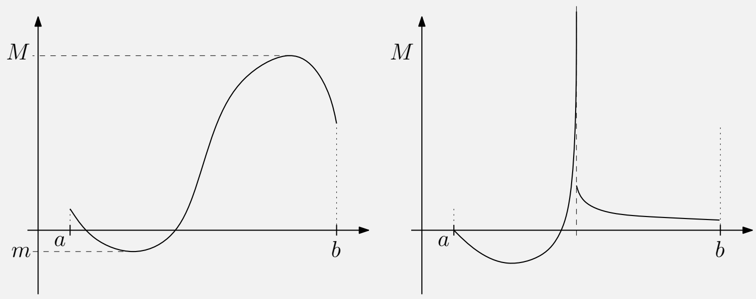
$$M = \sup\{f(x)/x \in [a,b]\} \qquad \text{and} \qquad m = \inf\{f(x)/x \in [a,b]\}$$

exist and are finite. Assume that f does not attain its maximum M then the function $\frac{1}{M-f(x)}$ will be defined and from the properties of continuity, it will be continuous over $[a,b]$. Following the same steps as in the first part of this proof, we can show that the function $\frac{1}{M-f(x)}$ is bounded. Therefore,

$$\forall x \in [a,b], \exists \alpha > 0, |M - f(x)| \geq \alpha.$$

In consequence, we have $\forall x \in [a,b], f(x) \leq M - \alpha$. This contradicts the fact that M is the least upper bound. Then necessarily f attains its maximum M. In the same way, we can establish that f attains its minimum m. ■

■ **Example 5.7**



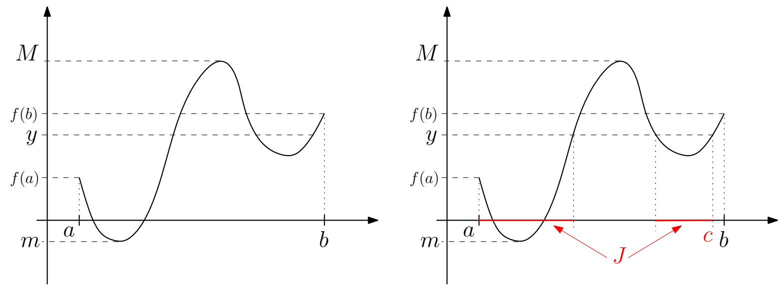
The function plotted on left fulfills the theorem's assumptions and clearly the function is bounded and its maximum and minimum values are attained (M and m). On the right, the function is discontinuous and in that case, the function does not attain its maximum value.

5.4.2 Intermediate value theorem

Theorem 5.4.2 — Intermediate Value Theorem
Let a function f continuous over an arbitrary interval I (it can be open, closed or semi-open; bounded or not). Denote $M = \sup f(I)$ and $m = \inf f(I)$ (note that these quantities can eventually be infinite). Then f takes all values in (m,M), i.e

$$\forall y \in (m,M), \exists x \in I, f(x) = y.$$

Proof. First note, that the Extreme Value Theorem cannot be used since I is not necessarily a closed bounded interval.
The case $f(x) = C$ (constant) is obvious since $m = M = C$.
Consider that $m \neq M$. From the properties of sup and inf, $\forall y \in (m,M), \exists a,b \in I, m \leq f(a) < y < f(b) \leq M$. Assume that $a < b$ (the case $b < a$ can be treated in a similar way).



Denote $J = \{x / f(x) \leq y\}$. Clearly, $J \subset [a,b]$ and $a \in J$ thus $J \neq \varnothing$ and is bounded above (b is an upper bound). Hence $c = \sup J$ exists.
Consider the sequence $\{x_n\}$ such that $\forall n \in \mathbb{N}, c - \frac{1}{n} \leq x_n \leq c$. We have

$$c = \lim_{n\to\infty}\left(c - \frac{1}{n}\right) \leq \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} c = c \Rightarrow \lim_{n\to\infty} x_n = c.$$

Since f is continuous, we have

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(c) \leq y.$$

On the other hand, $\forall x \in (c,b], f(x) > y$, then $\lim_{x\to c+} f(x) > \lim_{x\to c+} y = y$. Since f is continuous, we have that $\lim_{x\to c+} f(x) = \lim_{x\to c} f(x) = f(c)$ but $f(c)$ can eventually be equal to y thus we get $f(c) \geq y$ and finally $f(c) = y$. Therefore, we obtained that

$$\forall y \in (m,M), \exists x \in I, f(x) = y.$$

■

Corollary 5.4.3
The image of an arbitrary interval $I \subset \mathbb{R}$ by a continuous function f is an interval $J = f(I)$ of $\mathbb{R}$

Proof. Left as an exercise. Using the notation of the previous proof, show that J is either $[m,M], (m,M], [m,M)$ or (m,M). ■

Proposition 5.4.4
The image of a closed and bounded interval $I \subset \mathbb{R}$ by a continuous function is a closed and bounded interval.

Proof. It is a direct consequence of the extreme value theorem and intermediate value theorem. ■