6. Derivatives

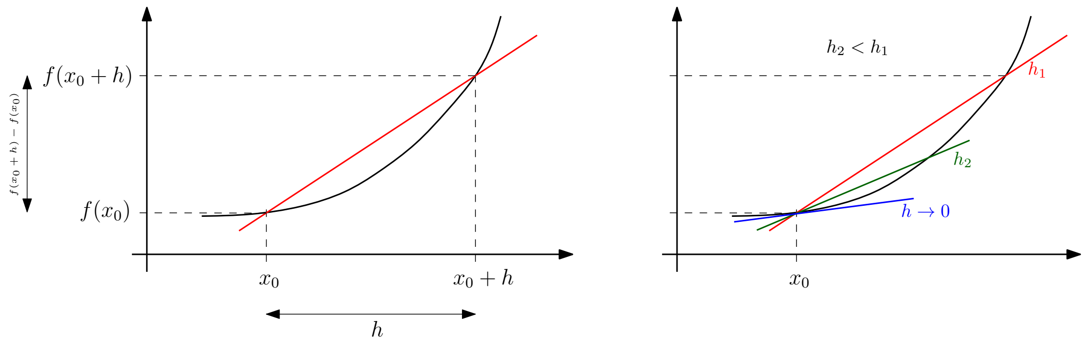
6.1 Derivative of a function

Definition 6.1.1
Let a function $f(x)$ defined over an open domain D. Let $x_0 \in D$. The **derivative** of f at x_0, and denoted $f'(x_0)$, is defined by

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h},$$

providing that the limit exists.

Geometric interpretation: the quantity $\frac{f(x_0)-f(x_0)}{h}$ corresponds to the slope of the chord passing through the points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$ (see left of figure below). When $h \to 0$, the behavior corresponds to the right of the figure, i.e the chord becomes the tangent line of f at x_0.



■ **Example 6.1**
Let $f(x) = 3x^2 + 4$, prove that $f'(1) = 6$.

Solution: We need to prove that

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=6.$$

We have,

$$\frac{f(1+h)-f(1)}{h}=\frac{(3(1+h)^2+4)-7}{h}=\frac{3(1+2h+h^2)-3}{h}=\frac{6h+3h^2}{h}=6+3h.$$

Therefore,

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}(6+3h)=6.$$

Definition 6.1.2
If the derivative of f at a point x_0 exists then f is said to be differentiable at x_0.

Proposition 6.1.1
Assume that f is differentiable at x_0 then f is continuous at x_0.

Proof. We have

$$f(x_0+h)-f(x_0)=\left(\frac{f(x_0+h)-f(x_0)}{h}\right)h,$$

then

$$f(x_0+h)=f(x_0)+\left(\frac{f(x_0+h)-f(x_0)}{h}\right)h.$$

Thus

$$\begin{aligned}\lim_{h\to 0}f(x_0+h)&=\lim_{h\to 0}f(x_0)+\lim_{h\to 0}\left(\left(\frac{f(x_0+h)-f(x_0)}{h}\right)h\right)\\&=\lim_{h\to 0}f(x_0)+\left(\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\right)\left(\lim_{h\to 0}h\right)\end{aligned}$$

Since f is differentiable, $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0)$ (finite). On the other hand $\lim_{h\to 0}f(x_0)=f(x_0)$ hence

$$\lim_{h\to 0}f(x_0+h)=f(x_0)+f'(x_0)\left(\lim_{h\to 0}h\right)=f(x_0)+f'(x_0)\times 0=f(x_0).$$

Therefore f is continuous at x_0. ■

R The fact that f is continuous does **NOT** provide that f is differentiable.
For instance, consider $f(x)=|x|$. We have $\lim_{x\to 0-}|x|=\lim_{x\to 0-}(-x)=0$ and $\lim_{x\to 0+}|x|=\lim_{x\to 0+}(x)=0$ thus $\lim_{x\to 0}|x|=0=f(0)\Leftrightarrow f$ is continuous at $x=0$.
On the other hand:

$$\lim_{h\to 0-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0-}\frac{|0+h|+|0|}{h}=\lim_{h\to 0-}\frac{|h|}{h}=\lim_{h\to 0-}\frac{-h}{h}=-1,$$

and

$$\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0+}\frac{|0+h|+|0|}{h}=\lim_{h\to 0+}\frac{|h|}{h}=\lim_{h\to 0+}\frac{h}{h}=1.$$

Therefore $\lim_{h\to 0+}\frac{f(0+h)-f(0)}{h}\neq\lim_{h\to 0-}\frac{f(0+h)-f(0)}{h}$, i.e $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist thus $f(x)=|x|$ is not differentiable at $x=0$.

Definition 6.1.3
Let a function $f(x): D \to R \subset \mathbb{R}$. The associated **derivative function** $f'(x): D' \to R' \subset \mathbb{R}$ is defined as

$$\forall x \in D', f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

where $D' \subseteq D$ is the set of points where the limit exists.

R It is usual to say "$f'(x)$ is the derivative of f".

Example 6.2
Let $f(x) = x^2$, find the derivative $f'(x)$.
Solution: we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h}$$
$$= \lim_{h \to 0} (2x + h) = 2x.$$

Notice that this calculus is defined $\forall x \in \mathbb{R}$ therefore $\forall x \in \mathbb{R}, f'(x) = 2x$.

6.2 Leibniz notation

Given a variable, for instance x, a "variation" of that variable is usually denoted by using Δ as a prefix to the variable, i.e Δx in our example. In the previous derivative definition, h represents a variation of x, so we can replace h by Δx. In the same way, $f(x+h) - f(x)$ represents the variation of f when we apply a variation $h = \Delta x$, so we can denote $\Delta f = f(x+h) - f(x)$. Therefore, the derivative definition can be restated as (Leibniz notation)

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df(x)}{dx}.$$

R Note that the notation $\frac{df}{dx}$ does **NOT** mean that we take the ratio of two quantities df and dx. The use of a lowercase d simply means that we consider a limit process.

Notation 6.1. *When we want to designate the derivative at a specific point x_0, we use the following notation:*

$$\left.\frac{df(x)}{dx}\right|_{x=x_0}$$

6.3 Properties of derivatives

Proposition 6.3.1 — Derivative of a constant.
Let $f(x) = c$ (constant) then $f'(x) = 0$.

Proof. We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

■

Proposition 6.3.2 — Power rule.
For all $x \in \mathbb{R}$ or eventually $x \in \mathbb{R}/\{0\}$, we have

$$\forall n \in \mathbb{Z}, (x^n)' = nx^{n-1}.$$

Proof. If $n = 0$, we have $\forall x \in \mathbb{R}, x^n = x^0 = 1$ then from the previous proposition, $(1)' = 0$ and the rule works as long we interpret $(0)(x^{-1}) = 0$.
Let $n \in \mathbb{Z}^+$, we have

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$

Using the binomial formula, we get

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n\right) - x^n}{h} \\
&= \lim_{h\to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}\right)}{h} \\
&= \lim_{h\to 0} \left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}\right) = nx^{n-1}.
\end{aligned}$$

The case $n \in \mathbb{Z}^-$ is equivalent to consider $n \in \mathbb{Z}^+$ and x^{-n}. Assume that $x \neq 0$, then we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^{-n} - x^{-n}}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^n} - \frac{1}{x^n}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^n - (x+h)^n}{(x+h)^n x^n}\right) \\
&= \lim_{h\to 0} \left[\left(-\frac{(x+h)^n - x^n}{h}\right)\left(\frac{1}{(x+h)^n x^n}\right)\right] \\
&= \left[-\lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h}\right)\right]\left[\lim_{h\to 0}\left(\frac{1}{(x+h)^n x^n}\right)\right] \\
&= (-nx^{n-1})\left(\frac{1}{x^{2n}}\right) = -nx^{-n-1}
\end{aligned}$$

■

Proposition 6.3.3 — Multiplication by a constant.
Let $c \in \mathbb{R}$ a constant and a function $f(x)$ differentiable at x, we have

$$(cf)'(x) = cf'(x).$$

Proof. See homework ■

Proposition 6.3.4 — Sum rule.
Let $f(x)$ and $g(x)$ differentiable at x. Then $f+g$ is differentiable at x and we have

$$(f+g)'(x) = f'(x) + g'(x).$$

Proof. See homework ■

Corollary 6.3.5
Let f and g be two functions differentiable at x and let $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$ two constants. Then $c_1 f + c_2 g$ is differentiable at x and we have

$$(c_1 f + c_2 g)'(x) = c_1 f'(x) + c_2 g'(x).$$

Proof. See homework ■

Proposition 6.3.6 — Product rule.
Let f and g be two differentiable functions at x. Then the product fg is differentiable at x and we have

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. See homework ■

Proposition 6.3.7 — Derivative of a reciprocal.
Let a differentiable function $f : D \to R \subset \mathbb{R}$ such that $\forall x \in D, f(x) \neq 0$. Then $1/f$ is also differentiable at x and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f^2(x)}.$$

Proof. We have

$$\begin{aligned}
\left(\frac{1}{f}\right)'(x) = \lim_{h\to 0} \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{f(x+h)} - \frac{1}{f(x)}\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\frac{f(x)-f(x+h)}{f(x+h)f(x)}\right) \\
&= \lim_{h\to 0} \left[\left(-\frac{f(x+h)-f(x)}{h}\right)\left(\frac{1}{f(x+h)f(x)}\right)\right] \\
&= \left(-\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right)\left(\lim_{h\to 0} \frac{1}{f(x+h)f(x)}\right)
\end{aligned}$$

But

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x),$$

and

$$\lim_{h\to 0} \frac{1}{f(x+h)f(x)} = \frac{1}{f^2(x)}.$$

Therefore

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{f(x)}.$$

■

Proposition 6.3.8 — Quotient rule.
Let f and g be two differentiable functions at x and assume that $g(x) \neq 0$. Then we have

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.$$

Proof. We use the product rule and the derivative of a reciprocal rule:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) = \left(f.\frac{1}{g}\right)'(x) &= f'(x)\frac{1}{g(x)} + f(x)\left(\frac{1}{g}\right)'(x)\\ &= f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right)\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}.\end{aligned}$$

■

6.4 Local linear approximations

We saw previously that the derivative of f at x_0 corresponds to the slope of the tangent line to f at x_0. Its equation is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Then we have the following definition.

Definition 6.4.1
The **linear approximation** of f at x_0 (sometimes called the **basepoint**) is given by

$$L_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0).$$

■ Example 6.3
Let $f(x) = 2x^2 + 5x - 3$. Find the linear approximation of f at 1.
Solution: We have $f'(x) = 4x + 5$ then $f'(1) = 9$ and $f(1) = 2 + 5 - 3 = 4$. Therefore

$$L_1(x) = 4 + 9(x - 1) = 9x - 5.$$

We can compute the absolute error between f and its approximation when we move away from x_0: $|f(1+h) - L_1(1+h)|$:

$$\begin{aligned}|f(1+h) - L_1(1+h)| &= |2(1+h)^2 + 5(1+h) - 3 - 9(1+h) + 5|\\ &= |2(1 + 2h + h^2) + 5 + 5h - 3 - 9 - 9h + 5| = 2h^2.\end{aligned}$$

Thus when we stay close to 1, the error stays small while if we move further from 1, it increases. This why it is a **local** approximation.
The function f and its approximation L_1 are illustrated in Figure 6.1 for different zooms.

Theorem 6.4.1
Let $f : D \to R \subset \mathbb{R}$ where D is an open interval. Let a point $x_0 \in D$. Let L be the line with slope

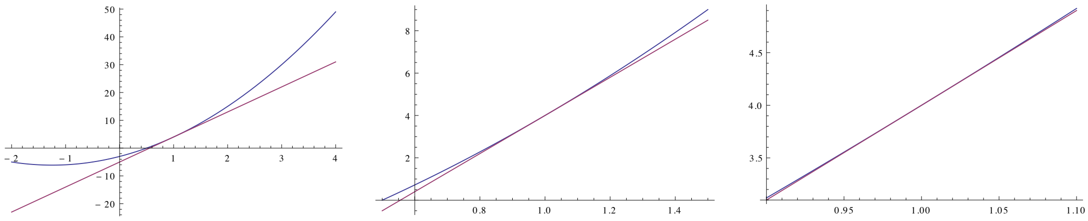


Figure 6.1: $f(x)$ and its local linear approximation $L_1(x)$ for different zooms.

m passing through $(x_0, f(x_0))$: $L(x) = f(x_0) + m(x - x_0)$. If

$$f(x_0+h) = L(x_0+h) + hQ_{x_0}(h)$$

where $\lim_{h\to 0} Q_{x_0}(h) = 0$ then the function f is differentiable at x_0 and $m = f'(x_0)$, i.e $L(x) = L_{x_0}(x)$.

Proof. We have $L(x) = f(x_0) + m(x - x_0)$, thus

$$L(x_0+h) = f(x_0) + m(x_0+h-x_0) = f(x_0) + mh.$$

Therefore,

$$f(x_0+h) - L(x_0+h) = f(x_0+h) - f(x_0) - mh.$$

If we denote

$$hQ_{x_0}(h) = f(x_0+h) - f(x_0) - mh,$$

we have

$$Q_{x_0}(h) = \frac{f(x_0+h) - f(x_0)}{h} - m.$$

Then,

$$\lim_{h\to 0} Q_{x_0}(h) = \lim_{h\to 0}\left(\frac{f(x_0+h) - f(x_0)}{h} - m\right).$$

Thus if $\lim_{h\to 0} Q_{x_0}(h) = 0$, we get $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = m$ since m is finite, the limit is finite thus f is differentiable and we have $m = f'(x_0)$ and $L(x) = L_{x_0}(x)$. ■

6.5 Higher-order derivatives

Since the derivative of f, f', is a function itself, we can consider the derivative of f'.

Definition 6.5.1
The second-order derivative of f is defined and denoted by

$$f''(x) = (f'(x))' \qquad \text{or} \qquad \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df(x)}{dx}\right).$$

Of course the domain of definition of f'' must be adapted in consequence.

The second-order derivative is also itself a function and we can consider its derivative, i.e the third-order derivative of f. This reasoning can be iterated as many time as we want and lead to the definition of the k-th order derivative of f.

Definition 6.5.2
The k-th order derivative of f is defined by taking k successively times the derivative of f and is denoted by

$$f^{(k)}(x) = (f^{(k-1)}(x))' = (\dots(f(x))'\dots)' \qquad \text{or} \qquad \frac{d^k f(x)}{dx^k} = \frac{d}{dx}\left(\dots\left(\frac{d}{dx}\left(\frac{df(x)}{dx}\right)\right)\dots\right).$$

Note the use of parenthesis in $f^{(k)}$ to distinguish between a k-th order and a k-th power.

6.6 The differential of a function and some of its consequences

We saw previously that we can rewrite the definition of a derivative as

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

If we consider $|\Delta x|$ small, an approximation of $f'(x)$ is given by

$$f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x},$$

$$\Leftrightarrow f(x+\Delta x)-f(x) \approx f'(x)\Delta x.$$

The left-hand side represents the increment of f when we add Δx to the basepoint x. Then the increment of f is a function of two variables: x and Δx. This quantity is called the differential of f:

Definition 6.6.1
The differential of f at the basepoint x and with increment Δx is denoted and defined by

$$df(x,\Delta x) = f'(x)\Delta x.$$

Therefore, if $|\Delta x|$ is small, we have

$$f(x+\Delta x)-f(x) \approx df(x,\Delta x).$$

Then we can restate Theorem 6.4.1 with this change of notation (the proof follows the same steps).

Theorem 6.6.1
Assume f is differentiable at x. Then

$$f(x+\Delta x)-f(x) = df(x,\Delta x) + \Delta x Q_x(\Delta x)$$

where $\lim_{\Delta x\to 0} Q_x(\Delta x) = 0$.

Notation 6.2. *Traditionally, Δx is denoted dx: $df(x,dx) = f'(x)dx = \frac{df(x)}{dx}dx = \frac{df}{dx}dx$.*

Example 6.4
Let $f(x) = \frac{1}{x^2}$. Express the differential of f and give an approximation of $f(2.1)$.
Solution: We have

$$df(x,dx) = f'(x)dx = -\frac{2}{x^3}dx.$$

On the other hand, we can write $2.1 = 2+0.1$, thus let $x = 2$ and $dx = 0.1$ thus

$$f(2.1) - f(2) \approx df(2, 0.1) = -\frac{2}{2^3} \times 0.1.$$

Therefore:

$$f(2.1) \approx f(2) - 0.025 = \frac{1}{4} - 0.025 = 0.25 - 0.025 = 0.225.$$

By a calculator, we get $f(2.1) \approx 0.22675737$ then our approximation makes an absolute error of $\approx 1.76.10^{-3}$.

The previous theorem helps to prove the well known chain rule to find the derivative of a composite function.

Theorem 6.6.2 — Chain rule
Assume that g is differentiable at x and f is differentiable at $g(x)$. Then $f \circ g$ is differentiable at x and we have

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Proof. By definition,

$$(f \circ g)'(x) = \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}.$$

Let denote $u = g(x)$ and $\Delta u = g(x+\Delta x) - g(x)$ thus $g(x+\Delta x) = u + \Delta u$. We have (using Theorem 6.6.1),

$$\begin{aligned} f(g(x+\Delta x)) - f(g(x)) = f(u+\Delta u) - f(u) &= df(u, \Delta u) + \Delta u Q_u(\Delta u) \\ &= f'(u)\Delta u + \Delta u Q_u(\Delta u), \end{aligned}$$

where $\lim_{\Delta u \to 0} Q_u(\Delta u) = 0$. Therefore,

$$\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \frac{f'(u)\Delta u + \Delta u Q_u(\Delta u)}{\Delta x} = f'(u)\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} Q_u(\Delta u),$$

and we get

$$\lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = \left(\lim_{\Delta x \to 0} f'(u)\right)\left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right) + \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)\left(\lim_{\Delta x \to 0} Q_u(\Delta u)\right).$$

Now, we address each term separately:

- $\lim_{\Delta x \to 0} f'(u) = \lim_{\Delta x \to 0} f'(g(x)) = f'(g(x))$ (since $f'(g(x))$ does not depends on Δx),
- $$\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = g'(x),$$
- since g is differentiable, it is continuous thus $\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0}(g(x+\Delta x) - g(x)) = g(x) - g(x) = 0$, i.e when $\Delta x \to 0$ we have $\Delta u \to 0$ as well. Therefore,

$$\lim_{\Delta x \to 0} Q_u(\Delta u) = \lim_{\Delta u \to 0} Q_u(\Delta u) = 0.$$

Using these results, we finally get

$$\lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} = f'(g(x))g'(x) + g'(x)(0) = f'(g(x))g'(x).$$

■

Another consequence is the formula to get the derivative of an inverse function.