Theorem 6.6.3
Assume the function f is either increasing or decreasing and differentiable at y. Let $y = f^{-1}(x)$ and $f'(y) \neq 0$. Then f^{-1} is differentiable at x and we have

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Proof. By definition,

$$\left(f^{-1}\right)'(x) = \lim_{\Delta x \to 0} \frac{f^{-1}(x+\Delta x) - f^{-1}(x)}{\Delta x}.$$

Let $y + \Delta y = f^{-1}(x + \Delta x)$, i.e $x + \Delta x = f(y + \Delta y)$. Thus, $\Delta x = f(y+\Delta y) - x = f(y+\Delta y) - f(y)$. Therefore,

$$\frac{f^{-1}(x+\Delta x) - f^{-1}(x)}{\Delta x} = \frac{(y+\Delta y) - y}{\Delta x} = \frac{\Delta y}{\Delta x}.$$

Notice that

$$\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(y+\Delta y) - f(y)}{\Delta y} = f'(y) \neq 0.$$

On the other hand, since f is differentiable, it is continuous thus $\lim_{\Delta y \to 0} \Delta x = \lim_{\Delta y \to 0}(f(y+\Delta y) - f(y) = f(y) - f(y) = 0$, i.e when $\Delta y \to 0$ we have $\Delta x \to 0$. Therefore,

$$\begin{aligned}\left(f^{-1}\right)'(x) = \lim_{\Delta x \to 0} \frac{f^{-1}(x+\Delta x) - f^{-1}(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\frac{\Delta x}{\Delta y}} \\ &= \frac{1}{\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y}} \\ &= \frac{1}{\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y}} \\ &= \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.\end{aligned}$$

■

■ **Example 6.5**
Find $\frac{d}{dx} \arcsin x$.
<u>Solution</u>: we have $\arcsin x = \sin^{-1}(x)$ thus

$$\frac{d}{dx} \arcsin x = \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\frac{d}{dy}(\sin y)\Big|_{y=\arcsin x}} = \frac{1}{\cos(\arcsin x)}$$

Since $\cos x = \sqrt{1 - \sin^2 x}$, we get

$$\frac{d}{dx} \arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}.$$

6.7 Mean Value Theorem

Theorem 6.7.1 — Fermat's theorem
If f has a local maximum or minimum at x_0 and f is differentiable at x_0 then we have $f'(x_0) = 0$.

Proof. Let us prove it when f has a local maximum. The minimum case can be proven in a similar way.
Since f is differentiable at x_0, we have

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h}.$$

Moreover, since f has a maximum at x_0, we have $f(x_0+h) \leq f(x_0)$, i.e $f(x_0+h)-f(x_0) \leq 0$, if $|h|$ is sufficiently small. Therefore,

- if $h > 0$, we have $\frac{f(x_0+h)-f(x_0)}{h} \leq 0 \Rightarrow \lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$.
- if $h < 0$, we have $\frac{f(x_0+h)-f(x_0)}{h} \geq 0 \Rightarrow \lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h} \geq 0$.

Thus

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0+} \frac{f(x_0+h)-f(x_0)}{h} \leq 0,$$

and

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0-} \frac{f(x_0+h)-f(x_0)}{h} \geq 0,$$

i.e $(f'(x_0) \leq 0) \wedge (f'(x_0) \geq 0) \Leftrightarrow f'(x_0) = 0$. ■

Theorem 6.7.2 — Rolle's theorem
Let f a function continuous on $[a,b]$ and differentiable in (a,b) such that $f(a) = f(b)$. Then $\exists c \in (a,b), f'(c) = 0$.

Proof. The case when f is constant on $[a,b]$ is obvious since $\forall x \in [a,b], f'(x) = 0$.
Assume now that f is not constant, continuous on $[a,b]$, differentiable on (a,b) such that $f(a) = f(b)$. By the extreme value theorem, we know that f attains a minimum and maximum on $[a,b]$. These values cannot be equal to $f(a) = f(b)$ at the same time otherwise it would mean that f is constant which contradicts the assumption. Therefore, if the maximum of $f(x) \neq f(a)$ (and $f(b)$), $\exists c \in (a,b)$ such that $f(c)$ is this maximum of f and from Fermat's theorem we have $f'(c) = 0$. The same reasoning holds if we consider the minimum of f. Therefore $\exists c \in (a,b), f'(c) = 0$. ■

Theorem 6.7.3 — Mean Value Theorem
Let f a function continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)(b-a).$$

Note that the previous equality can be rewritten as

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

the left-hand side corresponds to the slope of the line passing through points $(a, f(a))$ and $(b, f(b))$ while $f'(c)$ is the slope of the tangent line to f at the point $(c, f(c))$. The Mean value theorem means that there exists a tangent line to f which is parallel to the line passing through the endpoints of f.

Proof. Let denote g the secant line passing through $(a, f(a))$ and $(b, f(b))$. We know that the slope of this line is $\frac{f(b)-f(a)}{b-a}$ and the equation of g is given by

$$g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$

Now, we set

$$h(x) = f(x) - g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Clearly, h is continuous on $[a,b]$ and differentiable on (a,b) and we have $h(a) = h(b) = 0$. Therefore, by Rolle's theorem $\exists c \in (a,b), h'(c) = 0$. But

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a},$$

and the condition $h'(c) = 0$ is equivalent to

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \Leftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

■

Theorem 6.7.4 — Generalized (Cauchy's) Mean Value Theorem
Let two functions f and g continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c \in (a,b)$ such that

$$f'(c)[g(b)-g(a)] = g'(c)[f(b)-f(a)].$$

Proof. Set

$$h(x) = [f(x)-f(a)][g(b)-g(a)] - [f(b)-f(a)][g(x)-g(a)].$$

Clearly, h is continuous on $[a,b]$ and differentiable on (a,b) and $h(a) = h(b) = 0$. Therefore, Rolle's theorem holds and we get $\exists c \in (a,b), h'(c) = 0$. But,

$$h'(x) = f'(x)[g(b)-g(a)] - [f(b)-f(a)]g'(x),$$

thus the condition $h'(c) = 0$ is equivalent to

$$f'(c)[g(b)-g(a)] - [f(b)-f(a)]g'(c) = 0 \Leftrightarrow f'(c)[g(b)-g(a)] = [f(b)-f(a)]g'(c).$$

■

The Mean Value Theorem and its generalized version have several important consequences we will investigate now.

Theorem 6.7.5
Let a function f continuous on $[a,b]$ and differentiable on (a,b).

- If $\forall x \in (a,b), f'(x) > 0$ then f is increasing on $[a,b]$,
- If $\forall x \in (a,b), f'(x) < 0$ then f is decreasing on $[a,b]$

Proof. We have $\forall x_1, x_2 \in [a,b], x_1 < x_2$. We have $[x_1, x_2] \subset [a,b]$ and $(x_1, x_2) \subset [a,b]$. Thus, f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) and the Mean Value Theorem can be applied to f on $[x_1, x_2]$:

$$\exists c \in (x_1, x_2), f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Therefore

- if $\forall x \in [a,b], f'(x) > 0$ then $f'(c) > 0$ and we get that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0 \Leftrightarrow f(x_2) > f(x_1)$, i.e f is increasing,
- if $\forall x \in [a,b], f'(x) < 0$ then $f'(c) < 0$ and we get that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) < 0 \Leftrightarrow f(x_2) < f(x_1)$, i.e f is decreasing.

■

Proposition 6.7.6
Let f a function continuous on $[a,b]$ and differentiable on (a,b). Assume that $\forall x \in (a,b), f'(x) = 0$ then f is constant on $[a,b]$.

Proof. We have $\forall x_1, x_2 \in (a,b)$, f is continuous on $[x_1,x_2]$ and differentiable on (x_1,x_2) and the Mean Value Theorem can be applied to f on $[x_1,x_2]$:

$$\exists c \in (x_1,x_2), f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $\forall x \in [a,b], f'(x) = 0$, we have $f'(c) = 0$ thus $f(x_2) - f(x_1) = 0 \Leftrightarrow f(x_1) = f(x_2)$, i.e f is a constant function. ■

Corollary 6.7.7
Assume that $\forall x \in [a,b], f'(x) = g'(x)$ then $\exists C$ (constant) such that $\forall x \in [a,b], f(x) = g(x) + C$.

Proof. Denote $\forall x \in [a,b], h(x) = f(x) - g(x)$ then $h'(x) = f'(x) - g'(x) = 0$. From Proposition 6.7.6, h is a constant function, i.e $\exists C, h(x) = C$. Therefore, $f(x) - g(x) = C \Leftrightarrow f(x) = g(x) + C$. ■

6.8 Convexity

Definition 6.8.1
A function f is convex over $[a,b]$ if

$$\forall x,y \in [a,b], x < y, \forall \lambda \in [0,1], f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

A function f is concave over $[a,b]$ if

$$\forall x,y \in [a,b], x < y, \forall \lambda \in [0,1], f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y).$$

These definitions are illustrated in Figure 6.2.

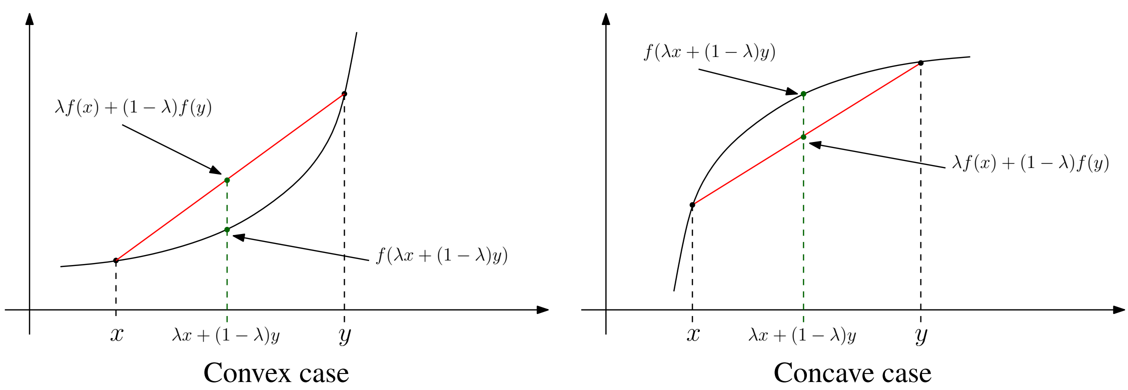

Figure 6.2: Example of convex and concave functions.

Proposition 6.8.1
A function f is a convex function on $[a,b]$ if and only if $\forall x,y \in [a,b], x<y$ and $\forall z \in [x,y]$, we have

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}.$$

Proof. $\Rightarrow$: Since $\forall x,y \in [a,b]$ we choose $z \in [x,y]$, we can write $z = \lambda x + (1-\lambda)y$ where $\lambda \in [0,1]$. If we solve for λ, we find

$$\lambda = \frac{y-z}{y-x} \qquad \text{and} \qquad 1-\lambda = \frac{z-x}{y-x}.$$

Since f is convex, we have

$$f(z) = f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Therefore, we have

$$f(z)-f(x) \leq \lambda f(x) + (1-\lambda)f(y) - f(x) = (1-\lambda)(f(y)-f(x)) = \frac{z-x}{y-x}(f(y)-f(x))$$

$$\Leftrightarrow \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(x)}{y-x}.$$

On the other hand, we have

$$-f(z) \geq -\lambda f(x) - (1-\lambda)f(y)$$

$$\Leftrightarrow f(y)-f(z) \geq f(y) - \lambda f(x) - (1-\lambda)f(y) = \lambda(f(y)-f(x)) = \frac{y-z}{y-x}(f(y)-f(x))$$

$$\Leftrightarrow \frac{f(y)-f(x)}{y-x} \leq \frac{f(y)-f(z)}{y-z}.$$

Therefore, combining the two obtained inequalities, we get the expected result:

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}.$$

$\Leftarrow$: Now assume that $\forall x,y \in [a,b]$ and $\forall z \in [x,y]$ and we have

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}.$$

As in the first part of the proof, we can write $z = \lambda x + (1-\lambda)y, \lambda \in [0,1]$ and then

$$\lambda = \frac{y-z}{y-x} \qquad \text{and} \qquad 1-\lambda = \frac{z-x}{y-x}.$$

Thus, we get

$$\frac{f(z)-f(x)}{(1-\lambda)(y-x)} \leq \frac{f(y)-f(z)}{\lambda(y-x)}$$
$$\Leftrightarrow \lambda(f(z)-f(x)) \leq (1-\lambda)(f(y)-f(z))$$
$$\Leftrightarrow f(z)(\lambda + (1-\lambda)) \leq \lambda f(x) + (1-\lambda)f(y)$$
$$\Leftrightarrow f(z) \leq \lambda f(x) + (1-\lambda)f(y)$$
$$\Leftrightarrow f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

i.e f is convex. ■