Theorem 6.8.2
Assume that f is differentiable on (a,b). Then f is convex on (a,b) if and only if f' in increasing on (a,b).

Proof. $\Rightarrow$: Assume that f is convex and differentiable on (a,b). Let $x,y \in (a,b), x<y$ then from Proposition 6.8.1, we have

$$\forall z \in [x,y], \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}.$$

If we set $z = x+h$ where $h>0$, we get

$$\frac{f(x+h)-f(x)}{x+h-x} \leq \frac{f(y)-f(x+h)}{y-x-h},$$

thus

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \leq \lim_{h\to 0}\frac{f(y)-f(x+h)}{y-x-h}$$
$$\Leftrightarrow f'(x) \leq \frac{f(y)-f(x)}{y-x}.$$

On the other hand, if we set $z = y+h$ where $h<0$, we get

$$\frac{f(y+h)-f(x)}{y+h-x} \leq \frac{f(y)-f(y+h)}{y-y-h} = \frac{f(y+h)-f(y)}{h}$$

thus

$$\lim_{h\to 0}\frac{f(y+h)-f(x)}{y+h-x} \leq \lim_{h\to 0}\frac{f(y+h)-f(y)}{h}$$
$$\Leftrightarrow \frac{f(y)-f(x)}{y-x} \leq f'(y).$$

Finally, we get $f'(x) \leq f'(y)$, i.e f' is increasing.
$\Leftarrow$: assume that f' is increasing on (a,b). Let $x,y \in (a,b), x<y$, from Proposition 6.8.1, it is sufficient to prove that

$$\forall z \in [x,y], \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z}.$$

But, $\forall z \in [x,y]$, by the Mean Value Theorem, $\exists \xi \in (x,z), f(z)-f(x) = f'(\xi)(z-x)$ and $\exists \eta \in (z,y), f(y)-f(z) = f'(\eta)(y-z)$. Since f' is increasing in (a,b), we have $f'(\xi) \leq f'(\eta)$

$$\Leftrightarrow \frac{f(z)-f(x)}{z-x} \leq \frac{f(y)-f(z)}{y-z},$$

i.e f is convex. ■

Corollary 6.8.3
Assume that $f''(x)$ exists and that $\forall x \in (a,b), f''(x) > 0$ then f is convex on (a,b).

Proof. Since $f''(x) = (f')'(x) > 0$, it means by Theorem 6.7.5 that f' is increasing and from Theorem 6.8.2, we get that f is convex on (a,b). ■

6.9 L'Hôspital's rule

L'Hôspital's rule is useful to find limits which are initially of indeterminate form like $0/0$ or ∞/∞.

Theorem 6.9.1 — L'Hôspital's rule
Let f and g two continuous and differentiable functions $\forall x$ in an open interval J which contains a point x_0 (with the possible exception of x_0 itself). Assume that $\forall x \in J, x \neq x_0, g(x) \neq 0, g'(x) \neq 0$. If $\lim_{x\to x_0} f(x) = 0 = \lim_{x\to x_0} g(x)$ or $\lim_{x\to x_0} f(x) = \pm\infty = \lim_{x\to x_0} g(x)$ where x_0 can be either finite or infinite, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

if the latter limit if either finite or infinite.

Proof. Let first show that the different cases reduced to the case $x \to 0+$, i.e assume that

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f'(x)}{g'(x)}$$

holds (we will prove it at the end) and assume that the latter limit is defined. Then, if

- $x \to 0-$, we have

$$\lim_{x\to 0-} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f(-x)}{g(-x)} = \lim_{x\to 0+} \frac{\frac{d}{dx}f(-x)}{\frac{d}{dx}g(-x)} = \lim_{x\to 0+} \frac{-f'(-x)}{-g'(-x)} = \lim_{x\to 0+} \frac{f'(-x)}{g'(-x)}$$
$$= \lim_{x\to 0-} \frac{f'(x)}{g'(x)},$$

- $x \to 0$, we split it in the two cases $x \to 0+$ and $x \to 0-$,
- $x \to x_0$ (finite and $\neq 0$), we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f(x-x_0)}{g(x-x_0)} = \lim_{x\to 0} \frac{\frac{d}{dx}f(x-x_0)}{\frac{d}{dx}g(x-x_0)} = \lim_{x\to 0} \frac{f'(x-x_0)}{g'(x-x_0)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)},$$

- $x \to +\infty$, we have

$$\lim_{x\to +\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f\left(\frac{1}{x}\right)}{g\left(\frac{1}{x}\right)} = \lim_{x\to 0+} \frac{\frac{d}{dx}f\left(\frac{1}{x}\right)}{\frac{d}{dx}g\left(\frac{1}{x}\right)} = \lim_{x\to 0+} \frac{-\frac{1}{x^2}f'\left(\frac{1}{x}\right)}{-\frac{1}{x^2}g'\left(\frac{1}{x}\right)} = \lim_{x\to 0+} \frac{f'\left(\frac{1}{x}\right)}{g'\left(\frac{1}{x}\right)}$$
$$= \lim_{x\to +\infty} \frac{f'(x)}{g'(x)},$$

- $x \to -\infty$, we have

$$\lim_{x\to -\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f\left(-\frac{1}{x}\right)}{g\left(-\frac{1}{x}\right)} = \lim_{x\to 0+} \frac{\frac{d}{dx}f\left(-\frac{1}{x}\right)}{\frac{d}{dx}g\left(-\frac{1}{x}\right)} = \lim_{x\to 0+} \frac{\frac{1}{x^2}f'\left(-\frac{1}{x}\right)}{\frac{1}{x^2}g'\left(-\frac{1}{x}\right)} = \lim_{x\to 0+} \frac{f'\left(-\frac{1}{x}\right)}{g'\left(-\frac{1}{x}\right)}$$
$$= \lim_{x\to -\infty} \frac{f'(x)}{g'(x)},$$

If $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \pm\infty$, it suffices to consider $\lim_{x\to x_0} \frac{g(x)}{f(x)}$ since $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{1}{\lim_{x\to x_0} \frac{g(x)}{f(x)}}$ and this returns us to the previous cases. It remains to prove that

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f'(x)}{g'(x)}.$$

Assume that $x \to 0+$ and that f and g are differentiable and $g'(x) \neq 0$ on an interval $(0,b]$ (where $b > 0$), so that f'/g' makes sense. Since f and g are differentiable on $(0,b]$, they are continuous on $(0,b]$. Moreover, assume that $\lim_{x\to 0+} f(x) = 0 = \lim_{x\to 0+} g(x)$ (the case $\lim_{x\to 0+} f(x) = \pm\infty =$

$\lim_{x\to 0+} g(x)$ can be handled in a similar way by considering the functions $F(x) = 1/f(x)$ and $G(x) = 1/g(x)$), we can extend the definition of f and g by $f(0) = 0$ and $g(0) = 0$. Therefore, by the Generalized Mean Value theorem, we get

$$\exists c \in (0,b), f'(c)[g(b)-g(0)] = g'(c)[f(b)-f(0)] \Leftrightarrow f'(c)g(b) = g'(c)f(b) \Leftrightarrow \frac{f(b)}{g(b)} = \frac{f'(c)}{g'(c)}.$$

Then

$$\lim_{b\to 0+} \frac{f(b)}{g(b)} = \lim_{b\to 0+} \frac{f'(c)}{g'(c)}.$$

But since $c \in (0,b)$, if $b \to 0+$, we have $c \to 0+$ thus we relabel b and c as x and consider $x \to 0+$ and we get

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{f'(x)}{g'(x)}.$$

■

6.10 Application to the study of functions

In this section, we explore how the previously taught material is useful to study the behavior of functions and let us to be able to sketch the graph of functions without any software. Basically, the idea consist in looking at the sign of the derivative and second-order derivative and find particular points (zero-crossings, local minimum and maximum, inflection points,...).

6.10.1 Example 1

Study the function $f(x) = x^3 - x$.
Solution: First, note that f is defined over $\mathbb{R}$.
Let us find the crossing points of f: we can notice that $f(x) = x^3 - x = x(x^2-1) = x(x-1)(x+1)$ thus f has three roots: $x = -1, x = 0$ and $x = 1$.
We now study the derivative of f: we have $\forall x \in \mathbb{R}, f'(x) = 3x^2 - 1$ thus

$$f'(x) = 0 \qquad \Leftrightarrow \qquad x = \frac{1}{\sqrt{3}} \quad \text{or} \quad x = -\frac{1}{\sqrt{3}}$$

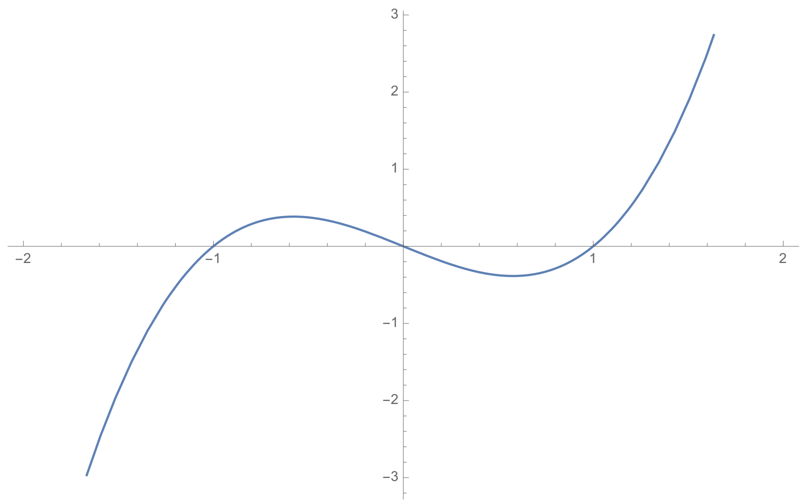
and we can easily check that $f'(x) < 0$ if $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $f'(x) > 0$ otherwise.
Next, we have $f''(x) = 6x$ thus the sign of f'' is the sign of x and f has an inflection point at 0. We can recap this information in the following table.

x	$-\infty$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$+\infty$
f''		$-$		0		$+$	
f'		$+$	0	$-$	0	$+$	
f	$-\infty$	0	$\frac{2}{3\sqrt{3}}$	0 inflection point	$-\frac{2}{3\sqrt{3}}$	0	$+\infty$

We easily check that

$$\lim_{x\to-\infty} f(x) = -\infty\ ;\ \lim_{x\to\infty} f(x) = +\infty\ ;\ f\left(-\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}} \approx 0.38\ ;\ f\left(\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}} \approx -0.38$$

The real graph of the function is given here



6.10.2 Example 2

Study the function $f(x) = \frac{e^x}{x-1}$.
Solution: First note that f is defined over $\mathbb{R}/\{1\}$ and clearly $\forall x \in \mathbb{R}/\{1\}, f(x) \neq 0$ and we have

$$\lim_{x\to-\infty} f(x) = 0\ ;\ \lim_{x\to\infty} f(x) = +\infty\ ;\ \lim_{x\to1^-} f(x) = -\infty\ ;\ \lim_{x\to1^+} f(x) = +\infty.$$

We now study the derivative of f:

$$f'(x) = \frac{e^x(x-1) - e^x(1)}{(x-1)^2} = e^x\frac{x-2}{(x-1)^2}.$$

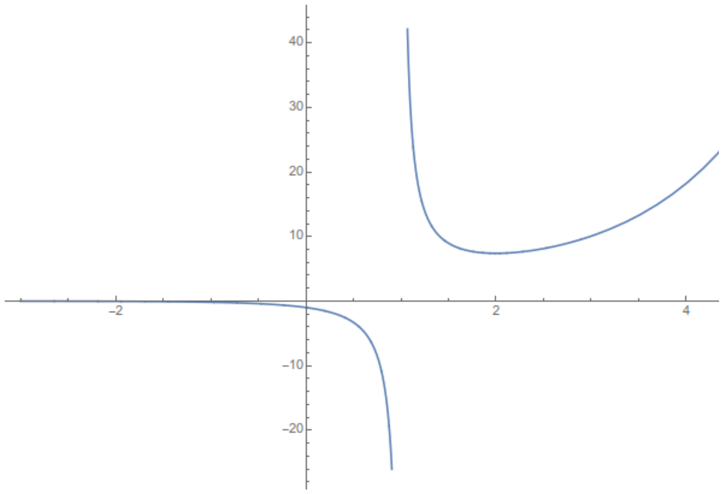
Hence $f'(x) = 0 \Leftrightarrow x = 2$ and $f(x) < 0$ if $x < 2$ and $x \neq 1$ and $f(x) > 0$ if $x > 2$.
Next, we have

$$f''(x) = e^x\frac{x-2}{(x-1)^2} + e^x\left(\frac{(x-1)^2 - (x-2)2(x-1)}{(x-1)^4}\right) = \frac{e^x}{(x-1)^3}(x^2 - 4x + 5).$$

It is easy to check that $x^2 - 4x + 5 > 0$ thus the sign of f'' is of the sign of $(x-1)^3$, i.e $f''(x) < 0$ if $x < 1$ and $f''(x) > 0$ if $x > 1$. This information is recap in the following table

x	$-\infty$		1		2		$+\infty$
f''		$-$	‖		$+$		
f'		$-$	‖	$-$	0	$+$	
f	0	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	e^2	$\nearrow$	$+\infty$

Obviously, $f(2) = e^2$ and $f(0) = -1$. The real graph of the function is given here



6.10.3 Example 3

Study the function $f(x) = e^{\sin x}$.
Solution: First, notice that f is defined over $\mathbb{R}$ and is $2\pi-$periodic thus is it sufficient to study the function on $[0, 2\pi]$.
Clearly, $\forall x \in [0, 2\pi], f(x) > 0$.
We now study the derivative of f: $f'(x) = e^{\sin x}\cos x$ thus

$$f'(x) = 0 \qquad \Leftrightarrow \qquad x = \frac{\pi}{2} \quad \text{or} \quad x = \frac{3\pi}{2}.$$

Moreover, we have $f'(x) < 0$ if $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and $f'(x) > 0$ otherwise.
Next, we have

$$f''(x) = e^{\sin x}\cos^2 x - e^{\sin x}\sin x = e^{\sin x}\left(1 - \sin^2 x - \sin x\right).$$

In order to solve $f''(x) = 0$, set $X = \sin x$ and we need to find X such that $1 - X^2 - X = 0$ which provides $X = \frac{-1\pm\sqrt{5}}{2}$ but since $X = \sin x$, we must have $X \in [-1, 1]$ thus the only solution is $X = \frac{-1+\sqrt{5}}{2}$. Therefore

$$\sin x = \frac{-1+\sqrt{5}}{2} \Leftrightarrow x_1 = \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 0.66 \quad \text{or} \quad x_2 = \pi - \arcsin\left(\frac{-1+\sqrt{5}}{2}\right) \approx 2.47.$$

We can easily check that $f''(x) < 0$ if $x \in (x_1, x_2)$ and $f''(x) > 0$ otherwise. We can recap this information in the following table.

x	0		x_1		$\frac{\pi}{2}$		x_2		$\frac{3\pi}{2}$		2π
f''		$+$	0		$-$		0		$+$		
f'			$+$		0		$-$		0	$+$	
f	1	↗			e	↘			e^{-1}	↗	1

Obviously, $f(\pi/2) = e$ and $f(2\pi/2) = e^{-1}$. The real graph, over three periods, of the function is given here