■ **Example 3.6**

Show that the sequence $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n=1}^{\infty}$ does NOT have a limit.
Solution: From the contraposition of the previous proposition, if we can exhibit two subsequences which have different limits then necessarily the original sequence does not have a limit.
Let's look at the first term of the sequence:

$$\sin\left(\frac{\pi}{2}\right);\sin(\pi);\sin\left(\frac{3\pi}{2}\right);\sin(2\pi);\sin\left(\frac{5\pi}{2}\right);\sin(3\pi);\sin\left(\frac{7\pi}{2}\right);\ldots$$

$$\Leftrightarrow 1;0;-1;0;1;0;-1;0;\ldots$$

Notice that $\sin\left((n+4)\frac{\pi}{2}\right)=\sin\left(\frac{n\pi}{2}+2\pi\right)=\sin\left(\frac{n\pi}{2}\right)$ thus the pattern $1,0,-1,0$ is repeated. Therefore, the sequences

$$1,1,1,1,\ldots$$

$$0,0,0,0,\ldots$$

$$-1,-1,-1,-1,\ldots$$

are subsequences of $\{a_n\}$ and have the limits $1,0,-1$, respectively. We finally conclude that the initial sequence does not have a limit.

3.3 Properties of limits

Proposition 3.3.1
The limit of a constant sequence c is c.

Proof. Assume that $\forall n\in\mathbb{N}, a_n=c$. We need to show that $\lim_{n\to\infty}a_n=c$.
We have that $\forall\varepsilon>0,\forall n\in\mathbb{N}$

$$|a_n-c|=|c-c|=0<\varepsilon.$$

Therefore $\lim_{n\to\infty}a_n=c$. ■

Proposition 3.3.2 — Constant sequence.
Assume that c is a constant and $\lim_{n\to\infty}a_n$ exists then

$$\lim_{n\to\infty}(ca_n)=c\lim_{n\to\infty}a_n.$$

Proof. If $c=0$ then $\forall n\in\mathbb{N}, ca_n=0$ so that $\lim_{n\to\infty}(ca_n)=\lim_{n\to\infty}(0)=0$ (from proposition 3.3.1).
If $c\neq 0$, denote $L=\lim_{n\to\infty}a_n$, then we have

$$\forall\varepsilon>0,\exists N\in\mathbb{N},\forall n\geq N,|a_n-L|<\frac{\varepsilon}{|c|},$$

$$\Leftrightarrow\forall\varepsilon>0,\exists N\in\mathbb{N},\forall n\geq N,|c||a_n-L|<\varepsilon,$$

$$\Leftrightarrow\forall\varepsilon>0,\exists N\in\mathbb{N},\forall n\geq N,|ca_n-cL|<\varepsilon.$$

Therefore

$$\lim_{n\to\infty}(ca_n)=cL=c\lim_{n\to\infty}a_n.$$

■

Proposition 3.3.3 — Sum rule.
Let $\{a_n\}$ and $\{b_n\}$ be two converging sequences then $\lim_{n\to\infty}(a_n+b_n)$ exists and

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

Proof. Since both $L_1 = \lim_{n\to\infty} a_n$ and $L_2 = \lim_{n\to\infty} b_n$ exist, we have $\forall \varepsilon > 0$

$$\exists N_1 \in \mathbb{N}, \forall n \geq N_1, |a_n - L_1| < \frac{\varepsilon}{2},$$

$$\exists N_2 \in \mathbb{N}, \forall n \geq N_2, |b_n - L_2| < \frac{\varepsilon}{2}.$$

Set $N = \max(N_1, N_2)$ then $\forall n \geq N$, we have (by the triangle inequality)

$$|(a_n+b_n)-(L_1+L_2)| = |(a_n-L_1)+(b_n-L_2)| \leq |a_n-L_1|+|b_n-L_2| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n.$$

∎

Definition 3.3.1
A sequence $\{a_n\}$ is said to be bounded if $\exists M \in \mathbb{R}, M \geq 0, \forall n \in \mathbb{N}, |a_n| \leq M$.

Proposition 3.3.4
A convergent sequence is bounded.

Proof. We need to show that $\exists M \geq 0, \forall n \in \mathbb{N}, |a_n| \leq M$.
Denote $L = \lim_{n\to\infty} a_n$ then

$$\exists N \in \mathbb{N}, \forall n \geq N, |a_n - L| < 1.$$

Therefore, if $n \geq N$ then

$$|a_n| = |a_n - L + L| \leq |a_n - L| + |L| < 1 + |L|.$$

Now, set $M = \max(|a_1|, |a_2|, \ldots, |a_{N-1}|, 1+|L|)$, we have $\exists M \geq 0, \forall n \in \mathbb{N}, |a_n| \leq M$. ∎

Proposition 3.3.5 — Product rule.
Let $\{a_n\}$ and $\{b_n\}$ be two converging sequences then

$$\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right).$$

Proof. Denote $L_1 = \lim_{n\to\infty} a_n$ and $L_2 = \lim_{n\to\infty} b_n$, we need to show that $\lim_{n\to\infty}(a_n b_n) = L_1 L_2$.
We have

$$\begin{aligned} |a_n b_n - L_1 L_2| &= |a_n b_n - L_1 b_n + L_1 b_n - L_1 L_2| \\ &= |(a_n - L_1) b_n + L_1 (b_n - L_2)| \\ &\leq |a_n - L_1||b_n| + |L_1||b_n - L_2|. \end{aligned}$$

Since a convergent sequence is bounded, $\exists M \geq 0$ such that $\forall n \in \mathbb{N}, |b_n| \leq M$. Therefore,

$$|a_n b_n - L_1 L_2| \leq M|a_n - L_1| + |L_1||b_n - L_2|.$$

Since $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$, we have $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$,

$$|a_n - L_1| < \frac{\varepsilon}{M + |L_1|} \qquad \text{and} \qquad |b_n - L_2| < \frac{\varepsilon}{M + |L_1|}.$$

Thus $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N$,

$$|a_n b_n - L_1 L_2| < M \frac{\varepsilon}{M + |L_1|} + |L_1| \frac{\varepsilon}{M + |L_1|} < \varepsilon \left(\frac{M}{M + |L_1|} + \frac{|L_1|}{M + |L_1|} \right) = \varepsilon.$$

Therefore

$$\lim_{n\to\infty} (a_n b_n) = L_1 L_2 = \left(\lim_{n\to\infty} a_n \right) \left(\lim_{n\to\infty} b_n \right).$$

■

Proposition 3.3.6 — Quotient rule.
Let $\{a_n\}$ and $\{b_n\}$ be two converging sequences. Assume that $\lim_{n\to\infty} b_n \neq 0$ then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

Proof. We only need to prove that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}$ because by product rule we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\lim_{n\to\infty} a_n \right) \left(\frac{1}{\lim_{n\to\infty} b_n} \right).$$

Denote $L = \lim_{n\to\infty} b_n$, let us prove that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{L}$. We have

$$\left| \frac{1}{b_n} - \frac{1}{L} \right| = \left| \frac{L - b_n}{b_n L} \right| = \frac{|b_n - L|}{|b_n||L|}.$$

Since $\lim_{n\to\infty} b_n = L$,

$$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, \forall n \geq N_1, |b_n - L| < \varepsilon \frac{L^2}{2},$$

and $\exists N_2 \in \mathbb{N}, \forall n \geq N_2$,

$$\begin{aligned} |b_n - L| < \frac{L}{2} &\Leftrightarrow -\frac{L}{2} < b_n - L < \frac{L}{2} \\ &\Leftrightarrow \frac{L}{2} < b_n < \frac{3L}{2} \\ &\Leftrightarrow \frac{2}{3L} < \frac{1}{b_n} < \frac{2}{L}. \end{aligned}$$

Setting $N = \max(N_1, N_2)$, we have $\forall n \geq N, |b_n - L| < \varepsilon \frac{L^2}{2}$ and $\frac{1}{|b_n|} < \frac{2}{|L|}$. Therefore,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \left| \frac{1}{b_n} - \frac{1}{L} \right| = \frac{|b_n - L|}{|b_n||L|} < \frac{2}{|L|} \frac{1}{|L|} \varepsilon \frac{L^2}{2} = \varepsilon.$$

Thus

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{\lim_{n\to\infty} b_n}.$$

■

Proposition 3.3.7
Let $\{a_n\}$ and $\{b_n\}$ be two converging sequences. If $\forall n \in \mathbb{N}, a_n < b_n$ then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

Proof. Denote $L_1 = \lim_{n\to\infty} a_n$ and $L_2 = \lim_{n\to\infty} b_n$, we need to show that $L_1 \le L_2$.
To do so, we will prove that $\forall \varepsilon > 0, L_1 < L_2 + \varepsilon$.
We have,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \ge N, |a_n - L_1| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |b_n - L_2| < \frac{L_2}{2}.$$

Therefore,

$$\begin{aligned} L_1 - L_2 &= (L_1 - a_n) + (a_n - b_n) + (b_n - L_2) \\ &\le |L_1 - a_n| - (b_n - a_n) + |b_n - L_2| \\ &\underset{\text{since}\, b_n > a_n}{<} |L_1 - a_n| + |b_n - L_2|. \end{aligned}$$

Thus

$$L_1 - L_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \Leftrightarrow L_1 < L_2 + \varepsilon.$$

■

Corollary 3.3.8
Assume that $\forall n \in \mathbb{N}, a_n < M$ and $\lim_{n\to\infty} a_n$ exists then

$$\lim_{n\to\infty} a_n \le M.$$

Proof. We use the previous proposition with $b_n = M$ (constant sequence) and the fact that $\lim_{n\to\infty} M = M$. ■

R We cannot claim the strict inequality $\lim_{n\to\infty} a_n < M$ if $\forall n \in \mathbb{N}, a_n < M$. Indeed, consider $a_n = 1 - \frac{1}{n}$, we have $\forall n \in \mathbb{N}, a_n < 1$ but $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right) = 1$!

3.4 Cauchy sequences

Definition 3.4.1
A sequence $\{a_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \in \mathbb{N}, n \ge N, m \ge N, |a_n - a_m| < \varepsilon.$$

If we set $m = n + k$ for $k \in \mathbb{N}$, we get the equivalent definition

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in, n \ge N, \forall k \in \mathbb{N}, |a_{n+k} - a_n| < \varepsilon.$$

Geometric interpretation: for any arbitrary n and m large enough ($\ge N$), we have $a_m - \varepsilon < a_n < a_m + \varepsilon$. This means that a_n and a_m are trapped within a small tube of radius ε, see Figure. 3.3.

Proposition 3.4.1
A convergent sequence is a Cauchy sequence.

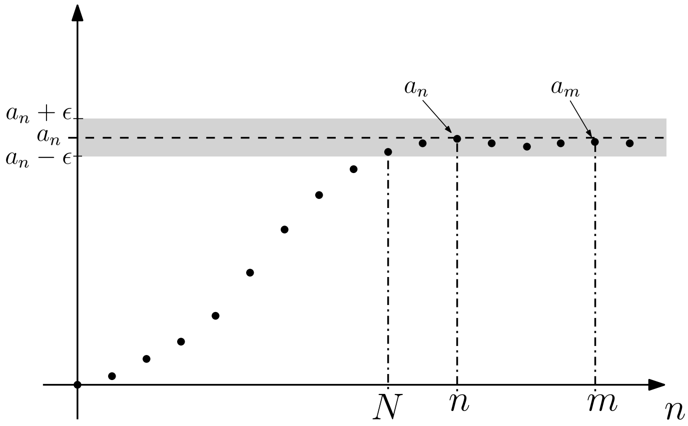

Figure 3.3: Geometric interpretation of a Cauchy sequence.

Proof. Let $\{a_n\}$ be a convergent sequence and denote L its limit. Then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |a_n - L| < \frac{\varepsilon}{2}.$$

Choose n and m such that $n \geq N$ and $m \geq N$. We have (by the triangle inequality)

$$|a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \in \mathbb{N}, n \geq N, m \geq N, |a_n - a_m| < \varepsilon,$$

i.e $\{a_n\}$ is a Cauchy sequence. ■

Definition 3.4.2
A **finite decimal** is an expression of the form

$$a_0.a_1a_2a_3\ldots a_n$$

where $a_0 \in \mathbb{Z}$ and $\forall n \in \mathbb{N}, a_n \in \{0,1,2,3,4,5,6,7,8,9\}$ (a_n are called digits). The corresponding rational number is

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \ldots + \frac{a_n}{10^n}.$$

An **infinite decimal** of the form

$$a_0.a_1a_2a_3\ldots a_n\ldots$$

corresponds to

$$\lim_{n\to\infty}\left(a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \ldots + \frac{a_n}{10^n}\right).$$

If a block of digits is repeated infinitely, then the limit is a rational number.

■ **Example 3.7**

$$\frac{1}{2} = 0.499999\ldots$$

$$\frac{1}{3} = 0.333333\ldots$$

Let us check that $1/2 = 0.499999\ldots$ (the second one is left as an exercise). We will use the identity

$$1+x+x^2+x^3+\ldots+x^n = \frac{1-x^{n+1}}{1-x} \qquad \text{if } x \neq 1.$$

We have

$$\begin{aligned}
\frac{4}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots+\frac{9}{10^n} &= \frac{4}{10}+\frac{9}{10^2}\left(1+\frac{1}{10}+\frac{1}{10^2}+\ldots+\frac{1}{10^{n-2}}\right)\\
&= \frac{4}{10}+\frac{9}{10^2}\frac{1-\frac{1}{10^{n-1}}}{1-\frac{1}{10}}\\
&= \frac{4}{10}+\frac{9}{10^2}\frac{1-\frac{1}{10^{n-1}}}{\frac{10-1}{10}}\\
&= \frac{4}{10}+\frac{1}{10}\left(1-\frac{1}{10^{n-1}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
0.49999\ldots &= \lim_{n\to\infty}\left(\frac{4}{10}+\frac{9}{10^2}+\frac{9}{10^3}+\ldots+\frac{9}{10^n}\right)\\
&= \lim_{n\to\infty}\left(\frac{4}{10}+\frac{1}{10}\left(1-\frac{1}{10^{n-1}}\right)\right)\\
&= \frac{4}{10}+\frac{1}{10}\lim_{n\to\infty}\left(1-\frac{1}{10^{n-1}}\right)\\
&= \frac{4}{10}+\frac{1}{10}=\frac{5}{10}=\frac{1}{2}.
\end{aligned}$$

Proposition 3.4.2

Given an infinite decimal

$$a_0.a_1a_2a_3\ldots a_n\ldots,$$

the sequence $\{S_n\}_{n=1}^{\infty}$ where

$$S_n = a_0+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots\frac{a_n}{10^n}$$

is a Cauchy sequence.

Proof. $\forall k \in \mathbb{N}$, we have

$$\begin{aligned}
|S_{n+k}-S_n| &= \Bigg|\left(a_0+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n}+\frac{a_{n+1}}{10^{n+1}}+\ldots+\frac{a_{n+k}}{10^{n+k}}\right)\\
&\qquad -\left(a_0+\frac{a_1}{10}+\frac{a_2}{10^2}+\ldots+\frac{a_n}{10^n}\right)\Bigg|\\
&= \left|\frac{a_{n+1}}{10^{n+1}}+\frac{a_{n+2}}{10^{n+2}}+\ldots+\frac{a_{n+k}}{10^{n+k}}\right|\\
&= \frac{a_{n+1}}{10^{n+1}}+\frac{a_{n+2}}{10^{n+2}}+\ldots+\frac{a_{n+k}}{10^{n+k}} \qquad \text{(each term in the sum is positive)}
\end{aligned}$$

But by definition, $\forall n \in \mathbb{N}, a_n \leq 9$ thus

$$
\begin{aligned}
|S_{n+k} - S_n| = \frac{a_{n+1}}{10^{n+1}} + \frac{a_{n+2}}{10^{n+2}} + \ldots + \frac{a_{n+k}}{10^{n+k}} &\leq \frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \ldots + \frac{9}{10^{n+k}} \\
&= \frac{9}{10^{n+1}} \left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^{k-1}}\right) \\
&= \frac{9}{10^{n+1}} \frac{1 - \frac{1}{10^k}}{1 - \frac{1}{10}} \\
&= \frac{9}{10^{n+1}} \frac{1 - \frac{1}{10^k}}{\frac{10-1}{10}} \\
&= \frac{1}{10^n} \left(1 - \frac{1}{10^k}\right) \\
&< \frac{1}{10^n}.
\end{aligned}
$$

Thus, $\forall \varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$\frac{1}{10^N} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 10^N \Leftrightarrow N > \log_{10}\left(\frac{1}{\varepsilon}\right).$$

Therefore,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \log_{10}\left(\frac{1}{\varepsilon}\right), \forall n \in \mathbb{N}, n \geq N, \forall k \in \mathbb{N}, |S_{n+k} - S_n| < \varepsilon,$$

which is the second form of the Cauchy sequence definition and we conclude that $\{S_n\}$ is a Cauchy sequence. ■

We know that a convergent sequence is a Cauchy sequence. It is then legitimate to ask if a Cauchy sequence is necessarily a convergent sequence. The answer is given by the **Cauchy convergence principle**:

Axiom 1
A Cauchy sequence of real numbers converges to a real number.

R Note that it concerns real numbers. For instance, this axiom does not hold for sequences of rational numbers!

R This axiom implies that any infinite decimal $a_0.a_1a_2a_3 \ldots a_n \ldots$ represents a real number because it is the limit of a Cauchy sequence.

R Assume we have shown that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence by determining an integer N_ε such that

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall m, n \in \mathbb{N}, n \geq N_\varepsilon, m \geq N_\varepsilon, |x_m - x_n| < \varepsilon.$$

In particular we have (because $n \geq N_\varepsilon$)):

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall m \in \mathbb{N}, m \geq N_\varepsilon, |x_m - x_{N_\varepsilon}| < \varepsilon.$$

If we denote $x = \lim_{m\to\infty} x_m$ then

$$|x - x_{N_\varepsilon}| = \lim_{m\to\infty} |x_m - x_{N_\varepsilon}| < \varepsilon.$$