But by definition, $\forall n \in \mathbb{N}, a_n \leq 9$ thus

$$\begin{aligned}
|S_{n+k} - S_n| = \frac{a_{n+1}}{10^{n+1}} + \frac{a_{n+2}}{10^{n+2}} + \ldots + \frac{a_{n+k}}{10^{n+k}} &\leq \frac{9}{10^{n+1}} + \frac{9}{10^{n+2}} + \ldots + \frac{9}{10^{n+k}} \\
&= \frac{9}{10^{n+1}}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^{k-1}}\right) \\
&= \frac{9}{10^{n+1}} \frac{1 - \frac{1}{10^k}}{1 - \frac{1}{10}} \\
&= \frac{9}{10^{n+1}} \frac{1 - \frac{1}{10^k}}{\frac{10-1}{10}} \\
&= \frac{1}{10^n}\left(1 - \frac{1}{10^k}\right) \\
&< \frac{1}{10^n}.
\end{aligned}$$

Thus, $\forall \varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$\frac{1}{10^N} < \varepsilon \Leftrightarrow \frac{1}{\varepsilon} < 10^N \Leftrightarrow N > \log_{10}\left(\frac{1}{\varepsilon}\right).$$

Therefore,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, N > \log_{10}\left(\frac{1}{\varepsilon}\right), \forall n \in \mathbb{N}, n \geq N, \forall k \in \mathbb{N}, |S_{n+k} - S_n| < \varepsilon,$$

which is the second form of the Cauchy sequence definition and we conclude that $\{S_n\}$ is a Cauchy sequence. ■

We know that a convergent sequence is a Cauchy sequence. It is then legitimate to ask if a Cauchy sequence is necessarily a convergent sequence. The answer is given by the **Cauchy convergence principle**:

Axiom 1
A Cauchy sequence of real numbers converges to a real number.

R Note that it concerns real numbers. For instance, this axiom does not hold for sequences of rational numbers!

R This axiom implies that any infinite decimal $a_0.a_1a_2a_3\ldots a_n\ldots$ represents a real number because it is the limit of a Cauchy sequence.

R Assume we have shown that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence by determining an integer N_ε such that

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall m, n \in \mathbb{N}, n \geq N_\varepsilon, m \geq N_\varepsilon, |x_m - x_n| < \varepsilon.$$

In particular we have (because $n \geq N_\varepsilon$)):

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall m \in \mathbb{N}, m \geq N_\varepsilon, |x_m - x_{N_\varepsilon}| < \varepsilon.$$

If we denote $x = \lim_{m\to\infty} x_m$ then

$$|x - x_{N_\varepsilon}| = \lim_{m\to\infty} |x_m - x_{N_\varepsilon}| < \varepsilon.$$

Therefore
$$x_{N_\varepsilon} - \varepsilon < x < x_{N_\varepsilon} + \varepsilon.$$
It means that we can approximate the number x with any arbitrary precision (even if we don't know the exact value of x, only x_{N_ε} is needed).

Proposition 3.4.3
Any real number x can be represented by a decimal.

Proof. Assume that $x > 0$ (the case $x < 0$ can be proven in a similar way), see Figure 3.4 for an illustration of this proof. By the Archimedean property, $\exists m \in \mathbb{N}, m > x$. By the well-ordering property of natural numbers, there exists a smallest m, we denote it $a_0 + 1$. We then have $a_0 \leq x < a_0 + 1$.
We divide the interval $[a_0, a_0 + 1)$ into ten subintervals of the same length:
$$[a_0, a_0+1) = \bigcup_{j=0}^{9} \left[a_0 + \frac{j}{10}, a_0 + \frac{j+1}{10}\right).$$
Note that all subintervals are disjoint then x belongs to only one of these subintervals. This is equivalent to
$$\exists a_1 \in \{0,1,2,3,4,5,6,7,8,9\}, x \in \left[a_0 + \frac{a_1}{10}, a_0 + \frac{a_1+1}{10}\right).$$
We can repeat the same process of dividing this subinterval into ten new subintervals:
$$\left[a_0 + \frac{a_1}{10}, a_0 + \frac{a_1+1}{10}\right) = \bigcup_{j=0}^{9} \left[a_0 + \frac{a_1}{10} + \frac{j}{10^2}, a_0 + \frac{a_1}{10} + \frac{j+1}{10^2}\right).$$
Hence,
$$\exists a_2 \in \{0,1,2,3,4,5,6,7,8,9\}, x \in \left[a_0 + \frac{a_1}{10} + \frac{a_2}{10^2}, a_0 + \frac{a_1}{10} + \frac{a_2+1}{10^2}\right).$$
We can repeat this process n times and create a sequence of intervals. Thus $\forall n \in \mathbb{N}$, $\exists a_n \in \{0,1,2,3,4,5,6,7,8,9\}$, such that
$$x \in \left[a_0 + \frac{a_1}{10} + \ldots + \frac{a_n}{10^n}, a_0 + \frac{a_1}{10} + \ldots + \frac{a_n+1}{10^n}\right)$$
or, if we denote
$$S_n = a_0 + \frac{a_1}{10} + \ldots + \frac{a_n}{10^n},$$
$$S_n \leq x \leq S_n + \frac{1}{10^n}.$$
We proved in Proposition 3.4.2, that the sequence $\{S_n\}$ is a Cauchy sequence. Therefore,
$$\exists y \in \mathbb{R}, \lim_{n\to\infty} S_n = y.$$
Since we have
$$y = \lim_{n\to\infty} S_n \leq x \leq \lim_{n\to\infty} S_n + \lim_{n\to\infty} \frac{1}{10^n} = y \Leftrightarrow x = y = \lim_{n\to\infty}\left(a_0 + \frac{a_1}{10} + \ldots + \frac{a_n}{10^n}\right)$$
and we conclude that x has a decimal representation. ∎

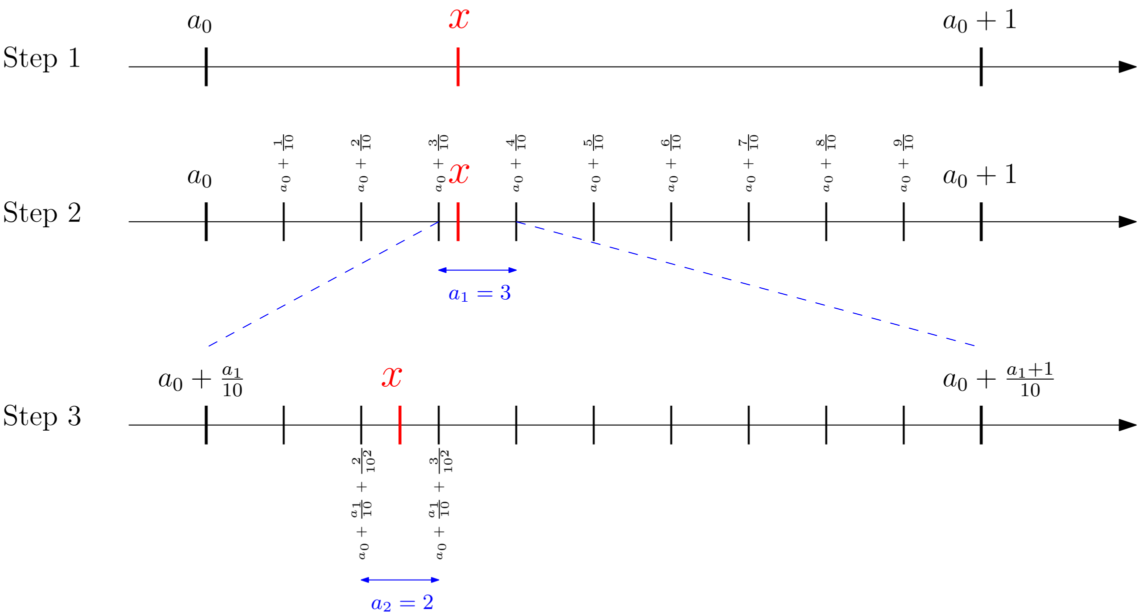


Figure 3.4: Construction of the sequence of disjoint subintervals.

Definition 3.4.3

A sequence of intervals $\{J_n\}_{n=1}^{\infty}$ is said to be a **nested sequence of intervals** if $\forall n \in \mathbb{N}, J_{n+1} \subset J_n$.

Theorem 3.4.4

Assume that $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a nested sequence of closed intervals such that

$$\lim_{n\to\infty}(b_n - a_n) = 0.$$

Then $\exists! x \in \mathbb{R}, \forall n \in \mathbb{N}, x \in [a_n, b_n]$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x.$$

Proof. Since $\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, [a_{n+k}, b_{n+k}] \subset [a_n, b_n]$, i.e $a_n \leq a_{n+k} \leq b_{n+k} \leq b_n$. Thus we have

$$|a_{n+k} - a_n| = a_{n+k} - a_n \leq b_n - a_n.$$

Since $\lim_{n\to\infty}(b_n - a_n) = 0$, we have

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, 0 < b_n - a_n < \varepsilon.$$

Therefore,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, \forall k \in \mathbb{N}, |a_{n+k} - a_n| < b_n - a_n < \varepsilon,$$

hence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore $\exists x \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = x$.
Similarly,

$$|b_{n+k} - b_n| = b_n - b_{n+k} \leq b_n - a_n < \varepsilon,$$

so that $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence as well. Therefore $\exists y \in \mathbb{R}$ such that $\lim_{n\to\infty} b_n = y$.
Now, we need to prove that $x = y$. We know that $\lim_{n\to\infty}(b_n - a_n) = 0$. But $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = y - x$ hence $x - y = 0 \Leftrightarrow x = y$.
The last step to prove is $\forall n \in \mathbb{N}, x \in [a_n, b_n]$. Since

$$\forall k \in \mathbb{N}, a_n \leq a_{n+k} \leq b_n$$

then

$$a_n \leq \lim_{k\to\infty} a_{n+k} = x \leq b_n.$$

Thus $\forall n \in \mathbb{N}, x \in [a_n, b_n]$. ■

This property is called the **completeness** of real numbers. The Cauchy convergence is one way of expressing completeness. The notion of completeness corresponds to the fact that the set of real numbers does not have any "hole" or there is no "gap" between consecutive numbers. It is clear that the set of rational numbers is not complete (we cannot represent all numbers by rational numbers, i.e there are some "gaps" between consecutive rational numbers). It is then easy to understand that the construction of real numbers made by completing the rational numbers with irrational numbers gives the completeness property of real numbers.

3.5 Countability

Definition 3.5.1
A set S is said **countable** if its elements can be listed as a sequence $a_1, a_2, a_3, \ldots$,

$$S = \{a_n / n \in \mathbb{N}\}.$$

Proposition 3.5.1
The set of rational numbers $\mathbb{Q}$ is countable.

Proof. We can list integers $\mathbb{Z}$ as

$$0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

thus we can list all possible rational numbers $\mathbb{Q}$ in the following way:

$$0, 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}, 3, -3, \frac{1}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}, 4, -4, \frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}, \ldots$$

■

R Even if the set of rational numbers is countable, there are infinitely many rational numbers in any interval. This property is characterized by the notion of density developed in the next section.

Proposition 3.5.2
The set $\mathbb{R}$ is uncountable.

Proof. We prove this proposition by contradiction: assume $\mathbb{R}$ is countable then $\exists\{a_n\}$ which represents all possible numbers of $\mathbb{R}$.
We have, $\forall n \in \mathbb{N}$, we can write the decimal expansion of a_n:

$$a_n = a_{n,0}.a_{n,1}a_{n,2}a_{n,3}\ldots a_{n,k}\ldots$$

Denote

$$b_m = \begin{cases} 0 & \text{if } a_{m,m} \neq 0 \\ 1 & \text{if } a_{m,m} = 0 \end{cases}.$$

We can build a real number b where its digits are given by the sequence $\{b_m\}$:

$$b = b_0.b_1b_2b_3\dots b_k\dots$$

(for example, if $a_0 = 1.20730; a_1 = 0.304; a_2 = 3.20863; a_3 = 0.83291\dots$ then $b_0 = 0; b_1 = 0; b_2 = 1; b_3 = 0; \dots$ and $b = 0.010\dots$).
We have $\forall n \in \mathbb{N}, b_n \neq a_n$ because by construction the nth digit of b is different of the nth digit of a_n. Thus we just built a real number $b \notin \{a_n\}$ thus $\{a_n\}$ do not represent all possible numbers of $\mathbb{R}$ and we get our contradiction so $\mathbb{R}$ must be uncountable. ■

Lemma 3.5.3
Let A and B be two countable sets then $A \cup B$ is a countable set.

Proof. Since A and B are countable there exists two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$A = \{a_n / n \in \mathbb{N}\} \qquad ; \qquad B = \{b_n / n \in \mathbb{N}\}.$$

Define the sequence $\{c_n\}$ such that $\forall k \in \mathbb{N}, c_{2k} = a_k$ and $c_{2k+1} = b_k$. Clearly the sequence $\{c_n\}$ contains all elements of both $\{a_n\}$ and $\{b_n\}$ which is equivalent to write

$$\{c_n / n \in \mathbb{N}\} = A \cup B.$$

Therefore $A \cup B$ is countable. ■

Proposition 3.5.4
The set of irrational numbers $\mathbb{R} \backslash \mathbb{Q}$ is uncountable.

Proof. We prove this proposition by contradiction: assume $\mathbb{R} \backslash \mathbb{Q}$ is countable then, since $\mathbb{Q}$ is countable and by Lemma 3.5.3, the set $\mathbb{Q} \cup (\mathbb{R} \backslash \mathbb{Q})$ is countable. But $\mathbb{Q} \cup (\mathbb{R} \backslash \mathbb{Q}) = \mathbb{R}$ which by Proposition 3.5.2 is uncountable. Then we get a contradiction and conclude that $\mathbb{R} \backslash \mathbb{Q}$ is uncountable. ■

3.6 Density

Definition 3.6.1
A subset $S \subset \mathbb{R}$ is **dense** in $\mathbb{R}$ if

$$\forall (a,b) \subset \mathbb{R}, \exists s \in S, s \in (a,b).$$

Proposition 3.6.1
The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$

$$\Leftrightarrow \forall a, b \in \mathbb{R}, a < b, \exists r \in \mathbb{Q}, r \in (a,b).$$

Proof. By the Archimedean property of $\mathbb{R}$, we know that

$$\forall a, b \in \mathbb{R}, a < b, \exists n \in \mathbb{N}, \frac{1}{n} < b - a.$$

Using again the Archimedean property of $\mathbb{R}$ and the well-ordering of positive integers, we know that there exists a smallest positive integer k such that

$$k - 1 \leq na \leq k \qquad \Leftrightarrow \qquad \frac{k-1}{n} \leq a \leq \frac{k}{n}.$$

Thus we have
$$a < \frac{k}{n} = \frac{k-1}{n} + \frac{1}{n} \leq a + \frac{1}{n} < a + (b-a) = b.$$
Therefore we found that there exists a rational number $r = \frac{k}{n}$ such that $\forall a, b \in \mathbb{R}, a < b, a < r < b$, this is equivalent to
$$\forall (a,b) \subset \mathbb{R}, \exists r \in \mathbb{Q}, r \in (a,b).$$
This is exactly the definition that $\mathbb{Q}$ is dense in $\mathbb{R}$. ■

Theorem 3.6.2
The set of irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.
$$\Leftrightarrow \forall a, b \in \mathbb{R}, a < b, \exists s \in (\mathbb{R}\backslash\mathbb{Q}), s \in (a,b).$$

Proof. We have $\forall a, b \in \mathbb{R}, a < b$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists q \in \mathbb{Q}, q \in \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.
Then
$$q\sqrt{2} \in (a,b).$$
Thus assume that $q' = q\sqrt{2} \in \mathbb{Q}$, since the ratio of two rational numbers is also a rational number, we get that $\sqrt{2} = \frac{q'}{q} \in \mathbb{Q}$ is a rational number, hence we reach a contradiction as we proved before that $\sqrt{2}$ is an irrational number (i.e $\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$).
Then necessarily, $q\sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$ and $q\sqrt{2} \in (a,b)$, i.e
$$\forall a, b \in \mathbb{R}, a < b, \exists s \in (\mathbb{R}\backslash\mathbb{Q}), s \in (a,b).$$
■

3.7 The least upper bound principle

Definition 3.7.1
Let $S \subset \mathbb{R}$.
- The number L is called the **least upper bound of** S if L is the smallest number such that $\forall x \in S, x \leq L$.
- The number l is the **greatest lower bound of** S if l is the greatest number such that $\forall x \in S, l \leq x$.

Notation 3.1. *The following notations are widely used in the literature:*
- *the least upper bound of S is also called the* ***supremum*** *of S and is denoted* $\sup S$,
- *the greatest lower bound of S is also called the* ***infimum*** *of S and is denoted* $\inf S$.

R Let a set S,
$$L = \sup S \Leftrightarrow (\forall x \in S, x \leq L) \wedge (\forall \varepsilon > 0, \exists x \in S, L - \varepsilon < x),$$
$$l = \inf S \Leftrightarrow (\forall x \in S, x \geq l) \wedge (\forall \varepsilon > 0, \exists x \in S, x < l + \varepsilon).$$
The statement $\forall x \in S, x \leq L$ corresponds to the fact that L is an upper bound of S. The statement $\forall \varepsilon > 0, \exists x \in S, L - \varepsilon < x$ corresponds to the fact that L is the least upper bound. Similar arguments can be used to justify the statement about infimum.

Proposition 3.7.1
If $L = \sup S$ and $l = \inf S$ then $\exists \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty}$ where $\forall n \in \mathbb{N}, x_n \in S$ and $y_n \in S$ and

$$\lim_{n\to\infty} x_n = L \qquad ; \qquad \lim_{n\to\infty} y_n = l.$$

Proof. Assume $L = \sup S$, from the previous remark, we have $\forall n \in \mathbb{N}, \exists x_n \in S$ such that

$$L - \frac{1}{n} < x_n \leq L$$

hence

$$0 \leq L - x_n < \frac{1}{n}.$$

Then $\lim_{n\to\infty} x_n = L$. A similar argument based on the second statement of the previous remark can be used to show that the existence of $\{y_n\}$ such that $\lim_{n\to\infty} y_n = l$. ■

R The least upper bound of a set S does not necessarily belong to S! For instance:

$$S = \left\{1 - \frac{1}{n} / n \in \mathbb{N}\right\},$$

then $\sup S = 1$ but $1 \notin S$. (The same type of argument holds for the greatest lower bound).
If $\sup S \in S$ then we say that $\sup S$ is the **maximum** value of numbers in S and we use the notation $\max S$. Similarly, if $\inf S \in S$ then $\inf S$ is the **minimum** value of numbers in S and we use the notation $\min S$.

Theorem 3.7.2 — The least upper bound principle

Let $S \subset \mathbb{R}, S \neq \varnothing$. We have
- if S is bounded above then $\sup S$ exists,
- if S is bounded below then $\inf S$ exists.

Proof. Let us prove the first statement. Assume that $S \neq \varnothing$ and S is bounded above. The idea consists in using Theorem 3.4.4 on nested sequence, to do so we need to build two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $\forall n \in \mathbb{N}, a_n \in S$ and b_n is an upper bound of S. We also need the following properties:

$$\forall n \in \mathbb{N}, a_n \leq a_{n+1} < b_{n+1} \leq b_n,$$

$$b_{n+1} - a_{n+1} \leq \frac{1}{2}(b_n - a_n).$$

First assume that such sequences exist (we will address their construction below).
From the definition of these sequences, it is clear, by construction, that the sequence of intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a sequence of nested intervals. Notice that

$$\forall n \in \mathbb{N}, 0 < b_n - a_n \leq \frac{1}{2^{n-1}}(b_1 - a_1) \qquad \text{and} \qquad \lim_{n\to\infty}(b_n - a_n) = 0.$$

Thus, by Theorem 3.4.4, $\exists! x \in \mathbb{R}, \forall n \in \mathbb{N}, x \in [a_n, b_n]$ and $\lim_{n\to\infty} a_n = lim_{n\to\infty} b_n = x$. We will prove that $x = \sup S$.
Since $\forall n \in \mathbb{N}, b_n$ are upper bounds of S, we have

$$\forall a \in S, \forall n \in \mathbb{N}, a \leq b_n,$$

Thus

$$a \leq \lim_{n\to\infty} b_n = x.$$

Therefore, x is an upper bound of S. On the other hand, since $\lim_{n\to\infty}(b_n - a_n) = 0$, we have

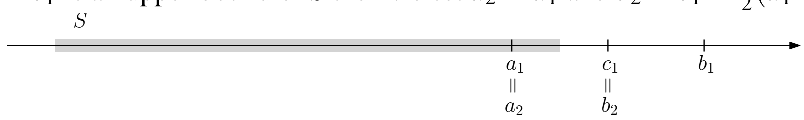
$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, b_N - a_N < \varepsilon.$$

But since $x \in [a_N, b_N]$, we have

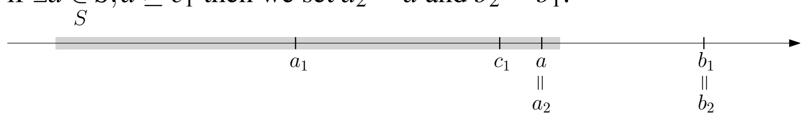
$$0 \leq x - a_N \leq b_N - a_N < \varepsilon.$$

Thus since $a_N \in S$ and $\forall \varepsilon > 0, a_N > x - \varepsilon$, x is the least upper bound of S.
It remains to show that it is actually possible to build such sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$.
Since $S \neq \varnothing$, $\exists a_1 \in S$ and since S is bounded above, S has an upper bound M then $\exists b_1 \geq M$ and b_1 is also an upper bound of S (i.e $b_1 > a_1$). Now, consider the midpoint of $[a_1, b_1]$: $c_1 = \frac{1}{2}(a_1 + b_1)$, we have two possible cases:

- if c_1 is an upper bound of S then we set $a_2 = a_1$ and $b_2 = c_1 = \frac{1}{2}(a_1 + b_1)$,



- if $\exists a \in S, a \geq c_1$ then we set $a_2 = a$ and $b_2 = b_1$.



We can iterate this process n times and get sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ with the two afore mentioned properties. Note that, by construction, we actually get

$$0 < b_n - a_n = \frac{1}{2^{n-1}}(b_1 - a_1) \qquad \text{and} \qquad \lim_{n\to\infty}(b_n - a_n) = 0$$

as desired.
The statement about the greatest lower bound can be established in a similar way. ■

3.8 Monotone sequences

Definition 3.8.1
A sequence $\{a_n\}$ is said to be

- **nondecreasing** or **increasing** if $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$,
- **strictly increasing** if $\forall n \in \mathbb{N}, a_n < a_{n+1}$,
- **nonincreasing** or **decreasing** if $\forall n \in \mathbb{N}, a_n \geq a_{n+1}$,
- **strictly decreasing** if $\forall n \in \mathbb{N}, a_n > a_{n+1}$,

In any of these case, we will say that $\{a_n\}$ is **monotone**.

Theorem 3.8.1 — Monotone convergence principle

A monotone increasing sequence $\{a_n\}$ of real numbers that is bounded above has a limit L and $\forall n \in \mathbb{N}, a_n \leq L$.
A monotone decreasing sequence $\{a_n\}$ of real numbers that is bounded below has a limit l and $\forall n \in \mathbb{N}, a_n \geq l$.

Proof. Assume that $\{a_n\}$ is an increasing sequence that is bounded above and denote $S = \{a_n / n \in \mathbb{N}\}$. By the least upper bound principle (Theorem 3.7.2), we have that $L = \sup S$ exists.