Show that

$$\lim_{x\to 2} 4(x+3) = 20.$$

We have $\forall x \in \mathbb{R}$,

$$|f(x)-20| = |4(x+3)-20| = 4|x-2|.$$

Therefore, in order to get $\forall \varepsilon > 0, |f(x)-20| = 4|x-2| < \varepsilon$, we can choose $|x-2| < \frac{\varepsilon}{4} = \delta$, thus

$$\forall \varepsilon > 0, \exists \delta > 0, \delta = \frac{\varepsilon}{4}, \forall x \in D, x \neq 2, |x-2| < \delta \Rightarrow |f(x)-20| < \varepsilon.$$

$$\Leftrightarrow \lim_{x\to 2} 4(x+3) = 20.$$

■ Example 4.2
Show that

$$\lim_{x\to -4} \frac{x^2-16}{10(x+4)} = -\frac{8}{10}.$$

Note that $D = \mathbb{R}/\{-4\}$, we have $\forall x \in D$,

$$\left| f(x) - \left(-\frac{8}{10}\right)\right| = \left|\frac{x^2-16}{10(x+4)} + \frac{8}{10}\right| = \frac{1}{10}\left|\frac{(x-4)(x+4)}{x+4} + 8\right| = \frac{1}{10}|(x-4)+8| = \frac{1}{10}|x+4|.$$

Therefore, in order to get $\forall \varepsilon > 0, \left|f(x) + \frac{8}{10}\right| < \varepsilon$, we can choose $|x+4| < 10\varepsilon = \delta$, thus

$$\forall \varepsilon > 0, \exists \delta > 0, \delta = 10\varepsilon, \forall x \in D, x \neq -4, |x+4| < \delta \Rightarrow \left|f(x) + \frac{8}{10}\right| < \varepsilon.$$

$$\Leftrightarrow \lim_{x\to -4} \frac{x^2-16}{10(x+4)} = -\frac{8}{10}.$$

Definition 4.1.2 — Alternative definition.
Given a function $f : D \to R \subset \mathbb{R}$, a point x_0 (eventually not in D) and $L \in \mathbb{R}$ (L being finite), we say that **the limit of the function f at x_0 is L** if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall h \in D, h \neq 0, |h| < \delta \Rightarrow |f(x_0+h) - L| < \varepsilon,$$

and we denote

$$\lim_{x\to x_0} f(x) = L.$$

■ Example 4.3
Use this definition in the first example above to show that

$$\lim_{x\to 2} 4(x+3) = 20.$$

We have $\forall h \in D$,

$$|f(2+h)-20| = |4(2+h+3)-20| = 4|h|.$$

Therefore, in order to get $\forall \varepsilon > 0, |f(2+h) - 20| = 4|h| < \varepsilon$, we can choose $|h| < \frac{\varepsilon}{4} = \delta$, thus

$$\forall \varepsilon > 0, \exists \delta > 0, \delta = \frac{\varepsilon}{4}, \forall h \in D, h \neq 0, |h| < \delta \Rightarrow |f(2+h) - 20| < \varepsilon.$$

$$\Leftrightarrow \lim_{x \to 2} 4(x+3) = 20.$$

Definition 4.1.3
Given a function $f : D \to R \subset \mathbb{R}$, a point x_0 (eventually not in D) and $L \in \mathbb{R}$ (L being finite), we say that **the limit from the right of the function f at x_0 is L** if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon,$$

and we denote

$$\lim_{x \to x_0+} f(x) = L.$$

Similarly we say that **the limit from the left of the function f at x_0 is L** if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon,$$

and we denote

$$\lim_{x \to x_0-} f(x) = L.$$

R The limit of $f(x)$ as x approaches x_0 exists $\Leftrightarrow$ both limits of $f(x)$ as x approaches x_0 from the right and from the left exist and are equal, i.e

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \left(\lim_{x \to x_0+} f(x) = L \right) \wedge \left(\lim_{x \to x_0-} f(x) = L \right).$$

Theorem 4.1.1 — Sequential characterization of limits
Given a function $f : D \to \mathbb{R}$ and a point x_0 (eventually not in D), then $\lim_{x \to x_0} f(x) = L \Leftrightarrow$ for any sequence $\{x_n\}$ where $\forall n \in \mathbb{N}, x_n \in D$ and where $\lim_{n \to \infty} x_n = x_0$, we have $\lim_{n \to \infty} f(x_n) = L$.

Proof. $\Rightarrow$: Assume

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Let an arbitrary sequence $\{x_n\}$ where $\forall n \in \mathbb{N}, x_n \in D$ and

$$\lim_{n \to \infty} x_n = x_0 \Leftrightarrow \forall \varepsilon', \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |x_n - x_0| < \varepsilon'.$$

In particular, for $\varepsilon' = \delta$ we have

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |x_n - x_0| < \delta.$$

By the assumption, this implies that $|f(x_n) - L| < \varepsilon$. Therefore

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N, |f(x_n) - L| < \varepsilon \Leftrightarrow \lim_{n \to \infty} f(x_n) = L.$$

$\Leftarrow$: Assume that for any sequence $\{x_n\}$ where $\forall n \in \mathbb{N}, x_n \in D$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} f(x_n) = L$.
We want to show that

$$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

We proceed by contradiction: assume

$$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in D, x \neq x_0, (|x - x_0| < \delta) \wedge (|f(x) - L| \geq \varepsilon).$$

But if we set $\delta = \frac{1}{n}$ then we can build a sequence $\{x_n\}$ such that $\forall n \in \mathbb{N}, \left(|x_n - x_0| < \frac{1}{n}\right) \wedge (|f(x_n) - L| \geq \varepsilon)$. Since $\lim_{n\to\infty} |x_n - x_0| < \lim_{n\to\infty} \frac{1}{n} = 0$, we get $\lim_{n\to\infty} x_n = x_0$. Thus we can build a sequence which is converging to x_0 and the sequence $\{f(x_n)\}$ does not converge to L which contradicts the initial assumption. Therefore

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon \Leftrightarrow \lim_{x\to x_0} f(x) = L.$$

■

Theorem 4.1.2
Assume that $f : [a,b] \to \mathbb{R}$ is an increasing or decreasing function. Then

$$\forall c \in (a,b), \lim_{x\to c-} f(x) \quad \text{and} \quad \lim_{x\to c+} f(x) \qquad \text{exist.}$$

In addition, if f is increasing, we have

$$\lim_{x\to c-} f(x) \leq f(c) \leq \lim_{x\to c+} f(x).$$

If f is decreasing, we have

$$\lim_{x\to c-} f(x) \geq f(c) \geq \lim_{x\to c+} f(x).$$

The limits $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist as well.

Proof. Consider the case when f is increasing and let $c \in (a,b)$. Let

$$S = \{f(x)/c < x \leq b\}.$$

Since f is increasing, $f(c)$ is a lower bound of S. Therefore, $L = \inf S$ exists.
We will prove that $\lim_{x\to c+} f(x) = L$.
Let an arbitrary $\varepsilon > 0$, by the definition of the greatest lower bound of a set, there exists $\delta > 0$ such that $c + \delta \leq b$ and

$$f(c) \leq f(c+\delta) < L + \varepsilon.$$

Since f is increasing, we have

$$\forall x, c < x < c+\delta, f(c) \leq f(x) \leq f(c+\delta) < L + \varepsilon.$$

Since L is a lower bound of the values $f(x)$ in $(c,b]$, we have

$$\forall x, c < x < c+\delta, L \leq f(x) \leq f(c+\delta) < L + \varepsilon \Leftrightarrow \lim_{x\to c+} f(x) = L \text{ exists.}$$

Since $\forall \varepsilon > 0, f(c) < L + \varepsilon$, we have $f(c) \leq L = \lim_{x\to c+} f(x)$.
To prove that $\lim_{x\to c-} f(x)$ exists and that $\lim_{x\to c-} f(x) \leq f(c)$, it is sufficient repeat the same steps by considering $\sup S$ where S is now defined by

$$S = \{f(x)/a \leq x < c\}.$$

The case of decreasing functions can be proven in a similar way. ■

Theorem 4.1.3 — Cauchy condition for the limit of a function
Assume that

- $f(x)$ is defined for $x \in (c, c+\delta_0)$, $\forall \delta_0 \in \mathbb{R}$. If
$$\forall \varepsilon > 0, \exists \delta > 0, \delta \leq \delta_0, \forall u, v \in \mathbb{R}, (c < u < c+\delta) \wedge (c < v < c+\delta) \Rightarrow |f(u) - f(v)| < \varepsilon.$$
Then $\lim_{x \to c+} f(x)$ exists.
- $f(x)$ is defined for $x \in (c-\delta_0, c)$, $\forall \delta_0 \in \mathbb{R}$. If
$$\forall \varepsilon > 0, \exists \delta > 0, \delta \leq \delta_0, \forall u, v \in \mathbb{R}, (c-\delta < u < c) \wedge (c-\delta < v < c) \Rightarrow |f(u) - f(v)| < \varepsilon.$$
Then $\lim_{x \to c-} f(x)$ exists.

Proof. We prove only the first statement, the second one by be proven in a similar way.
We know that

$$\forall \delta_0 \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \delta \leq \delta_0, \forall u, v \in \mathbb{R}, (c < u < c+\delta) \wedge (c < v < c+\delta) \Rightarrow |f(u) - f(v)| < \varepsilon.$$

In particular, for $\delta_0 = 1$ and $\varepsilon = 1$,

$$\exists \delta_1 > 0, \delta_1 \leq 1, \forall u, v \in \mathbb{R}, (c < u < c+\delta_1) \wedge (c < v < c+\delta_1) \Rightarrow |f(u) - f(v)| < 1.$$

Choosing x_1 such that $c < x_1 < c+\delta_1 < c+1$ and if $c < u < c+\delta_1$ we get

$$|f(u) - f(x_1)| < 1.$$

In the same way, for $\delta_0 = \min(1/2, \delta_1), \varepsilon = \frac{1}{2}$:
$\exists \delta_2 < \min(1/2, \delta_1)$ such that we can select a point x_2 where $c < x_2 < c+\delta_2 < c+\frac{1}{2}$ and if $c < u < c+\delta_2$ we have

$$|f(u) - f(x_2)| < \frac{1}{2}.$$

Notice that

$$|f(x_1) - f(x_2)| < 1.$$

We can iterate this process n times and build two sequences $x_1 > x_2 > x_3 > \ldots > x_n$ and $\delta_1 > \delta_2 > \delta_3 > \ldots > \delta_n$ such that

$$c < x_k < c+\delta_k < c+\frac{1}{k},$$

and if $c < u < c+\delta_k$ for $k = 1, 2, \ldots, n$ we have

$$|f(u) - f(x_k)| < \frac{1}{k}.$$

Therefore, we have

$$\lim_{n \to \infty} x_n = c.$$

Notice that

$$|f(x_{n+k}) - f(x_n)| < \frac{1}{n}.$$

Thus, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \frac{1}{N} < \varepsilon$ and $\forall n \in \mathbb{N}, n \geq N, \forall k \in \mathbb{N}$ we have

$$|f(x_{n+k}) - f(x_n)| < \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

Therefore, $\{f(x_n)\}$ is a Cauchy sequence and hence converges, i.e $\exists L \in \mathbb{R}, \lim_{n\to\infty} f(x_n) = L$. We will prove now that $\lim_{x\to c+} f(x) = L$. We have $\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}, N_1 > 2/\varepsilon$. Thus, if $\forall x, c < x < c + \delta_{N_1}$ we have $\forall n \in \mathbb{N}$,

$$|f(x) - L| \leq |f(x) - f(x_n)| + |f(x_n) - L|.$$

If we choose $n \geq N_1$ we get

$$|f(x) - L| \leq \frac{1}{N_1} + |f(x_n) - L| < \frac{\varepsilon}{2} + |f(x_n) - L|.$$

Since $\lim_{n\to\infty} f(x_n) = L$,

$$\forall \varepsilon > 0, \exists N_2 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq N_2, |f(x_n) - L| < \frac{\varepsilon}{2}.$$

Set $N = \max(N_1, N_2)$ we obtain

$$\forall \varepsilon > 0, \exists \delta_N \in \mathbb{R}, \forall x, c < x < c + \delta_N \Rightarrow |f(x) - L| < \varepsilon.$$

$$\Leftrightarrow \lim_{x\to c+} f(x) = L.$$

■

4.2 Infinite limits

Definition 4.2.1

Let a function $f : D \to R \subset \mathbb{R}$.

- $\lim_{x\to x_0} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta \Rightarrow f(x) > M$.
- $\lim_{x\to x_0+} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in D, x_0 < x < x_0 + \delta \Rightarrow f(x) > M$.
- $\lim_{x\to x_0-} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in D, x_0 - \delta < x < x_0 \Rightarrow f(x) > M$.
- $\lim_{x\to x_0} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta \Rightarrow f(x) < -M$.
- $\lim_{x\to x_0+} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in D, x_0 < x < x_0 + \delta \Rightarrow f(x) < -M$.
- $\lim_{x\to x_0-} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in D, x_0 - \delta < x < x_0 \Rightarrow f(x) < -M$.

R It is easy to check that

$$\lim_{x\to x_0} f(x) = -\infty \Leftrightarrow lim_{x\to x_0}(-f(x)) = +\infty.$$

■ **Example 4.4**

Let

$$f(x) = \frac{1}{(x+3)(x-2)}.$$

Prove that $\lim_{x\to 2+} f(x) = +\infty$ and $\lim_{x\to 2-} f(x) = -\infty$

<u>Solution</u>: we first address the case $\lim_{x\to 2+} f(x) = +\infty$:

We can restrict x such that $2 < x < 3$, thus $5 < x + 3 < 6$ and $\frac{1}{x+3} > \frac{1}{6}$. Therefore

$$f(x) = \frac{1}{(x+3)(x-2)} > \frac{1}{6(x-2)}.$$

In order to have $\forall M > 0, f(x) > M$ it is sufficient to have

$$\frac{1}{6(x-2)} > M \Leftrightarrow 0 < x - 2 < \frac{1}{6M} \Leftrightarrow 2 < x < 2 + \frac{1}{6M}.$$

Keeping in mind the constraint $x < 3 = 2+1$, we can set $\delta = \min(1, \frac{1}{6M})$. Thus we obtain,

$$\forall M > 0, \exists \delta > 0, \delta = \min\left(1, \frac{1}{6M}\right), \forall x \in D, 2 < x < 2+\delta \Rightarrow f(x) > M$$

$$\Leftrightarrow \lim_{x \to 2+} f(x) = +\infty.$$

For the second case, $\lim_{x \to 2-} f(x) = -\infty$, it is generally easier to prove $\lim_{x \to 2-}(-f(x)) = +\infty$. We have $x < 2$ thus $x+3 < 5$ and $\frac{1}{x+3} > \frac{1}{5}$. Therefore

$$-f(x) = \frac{1}{(x+3)(2-x)} > \frac{1}{5(2-x)}.$$

In order to have $\forall M > 0, -f(x) > M$ it is sufficient to have

$$\frac{1}{5(2-x)} > M \Leftrightarrow 0 < 2-x < \frac{1}{5M} \Leftrightarrow 2 - \frac{1}{5M} < x < 2.$$

If we set $\delta = \frac{1}{5M}$, we finally get

$$\forall M > 0, \exists \delta > 0, \delta = \frac{1}{5M}, \forall x \in D, 2-\delta < x < 2 \Rightarrow -f(x) > M$$

$$\Leftrightarrow \lim_{x \to 2-} f(x) = -\infty.$$

Proposition 4.2.1

Assume that $\forall x \in (a,b), f(x) > 0$, let $x_0 \in (a,b)$ such that $\lim_{x \to x_0} f(x) = 0$. Then

$$\lim_{x \to x_0} \frac{1}{f(x)} = +\infty.$$

Assume that $\forall x \in (a,b), f(x) < 0$, let $x_0 \in (a,b)$ such that $\lim_{x \to x_0} f(x) = 0$. Then

$$\lim_{x \to x_0} \frac{1}{f(x)} = -\infty.$$

Proof. We prove only the first statement, the second one can be proven in a similar way. We follow a similar argument as in the proof of Proposition 3.9.2. Since $\forall x \in (a,b), f(x) > 0, x_0 \in (a,b)$ and $\lim_{x \to x_0} f(x) = 0$, we have

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (a,b), x \neq x_0, |x - x_0| < \delta \Rightarrow |f(x) - 0| < \varepsilon.$$

Denote $M = \frac{1}{\varepsilon}$, then we have

$$\forall M > 0, \exists \delta > 0, \forall x \in (a,b), x \neq x_0, |x - x_0| < \delta \Rightarrow |f(x) - 0| < \frac{1}{M}.$$

$$\Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in (a,b), x \neq x_0, |x - x_0| < \delta \Rightarrow \frac{1}{f(x)} > M.$$

$$\Leftrightarrow \lim_{x \to x_0} \frac{1}{f(x)} = +\infty.$$

■