Keeping in mind the constraint $x < 3 = 2+1$, we can set $\delta = \min(1, \frac{1}{6M})$. Thus we obtain,

$$\forall M > 0, \exists \delta > 0, \delta = \min\left(1, \frac{1}{6M}\right), \forall x \in D, 2 < x < 2+\delta \Rightarrow f(x) > M$$

$$\Leftrightarrow \lim_{x \to 2+} f(x) = +\infty.$$

For the second case, $\lim_{x\to 2-} f(x) = -\infty$, it is generally easier to prove $\lim_{x\to 2-}(-f(x)) = +\infty$. We have $x < 2$ thus $x+3 < 5$ and $\frac{1}{x+3} > \frac{1}{5}$. Therefore

$$-f(x) = \frac{1}{(x+3)(2-x)} > \frac{1}{5(2-x)}.$$

In order to have $\forall M > 0, -f(x) > M$ it is sufficient to have

$$\frac{1}{5(2-x)} > M \Leftrightarrow 0 < 2-x < \frac{1}{5M} \Leftrightarrow 2 - \frac{1}{5M} < x < 2.$$

If we set $\delta = \frac{1}{5M}$, we finally get

$$\forall M > 0, \exists \delta > 0, \delta = \frac{1}{5M}, \forall x \in D, 2-\delta < x < 2 \Rightarrow -f(x) > M$$

$$\Leftrightarrow \lim_{x \to 2-} f(x) = -\infty.$$

Proposition 4.2.1
Assume that $\forall x \in (a,b), f(x) > 0$, let $x_0 \in (a,b)$ such that $\lim_{x\to x_0} f(x) = 0$. Then

$$\lim_{x\to x_0} \frac{1}{f(x)} = +\infty.$$

Assume that $\forall x \in (a,b), f(x) < 0$, let $x_0 \in (a,b)$ such that $\lim_{x\to x_0} f(x) = 0$. Then

$$\lim_{x\to x_0} \frac{1}{f(x)} = -\infty.$$

Proof. We prove only the first statement, the second one can be proven in a similar way. We follow a similar argument as in the proof of Proposition 3.9.2. Since $\forall x \in (a,b), f(x) > 0, x_0 \in (a,b)$ and $\lim_{x\to x_0} f(x) = 0$, we have

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (a,b), x \neq x_0, |x-x_0| < \delta \Rightarrow |f(x) - 0| < \varepsilon.$$

Denote $M = \frac{1}{\varepsilon}$, then we have

$$\forall M > 0, \exists \delta > 0, \forall x \in (a,b), x \neq x_0, |x-x_0| < \delta \Rightarrow |f(x) - 0| < \frac{1}{M}.$$

$$\Leftrightarrow \forall M > 0, \exists \delta > 0, \forall x \in (a,b), x \neq x_0, |x-x_0| < \delta \Rightarrow \frac{1}{f(x)} > M.$$

$$\Leftrightarrow \lim_{x\to x_0} \frac{1}{f(x)} = +\infty.$$

■

R The same result holds for left and right limits.

Example 4.5
Determine $\lim_{x\to\frac{\pi}{2}\pm}\sec(x)$.
Solution: We have $\sec(x)=\frac{1}{\cos(x)}$.

- If $0<x<\pi/2$ then $\cos(x)>0$ and $\lim_{x\to\pi/2-}\cos(x)=0$. Therefore
$$\lim_{x\to\pi/2-}\sec(x)=\lim_{x\to\pi/2-}\frac{1}{\cos(x)}=+\infty.$$
- If $\pi/2<x<3\pi/2$ then $\cos(x)<0$ and $\lim_{x\to\pi/2+}\cos(x)=0$. Therefore
$$\lim_{x\to\pi/2+}\sec(x)=\lim_{x\to\pi/2+}\frac{1}{\cos(x)}=-\infty.$$

Proposition 4.2.2
Assume that $\lim_{x\to x_0}f(x)=L>0$ (finite) or $\lim_{x\to x_0}f(x)=+\infty$ and $\lim_{x\to x_0}g(x)=+\infty$. Then
$$\lim_{x\to x_0}f(x)g(x)=+\infty.$$

Proof. First, assume that $\lim_{x\to x_0}f(x)=L>0$. Then
$$\forall\varepsilon>0,\exists\delta>0,\forall x\in D,x\neq x_0,|x-x_0|<\delta\Rightarrow|f(x)-L|<\varepsilon.$$
In particular, if $\varepsilon=\frac{L}{2}$:
$$\exists\delta_1>0,\forall x\in D,x\neq x_0,|x-x_0|<\delta_1\Rightarrow|f(x)-L|<\frac{L}{2}.$$
Then
$$f(x)-L>-\frac{L}{2}\Leftrightarrow f(x)>L-\frac{L}{2}=\frac{L}{2}.$$
On the other hand, since $\lim_{x\to x_0}g(x)=+\infty$, we have
$$\forall M>0,\exists\delta_2>0,\forall x\in D,x\neq x_0,|x-x_0|<\delta_2\Rightarrow g(x)>\frac{2M}{L}.$$
Set $\delta=\min(\delta_1,\delta_2)$ then we have
$$f(x)g(x)>\left(\frac{L}{2}\right)\left(\frac{2M}{L}\right)=M.$$
Therefore
$$\forall M>0,\exists\delta>0,\forall x\in D,x\neq x_0,|x-x_0|<\delta\Rightarrow f(x)g(x)>M\Leftrightarrow\lim_{x\to x_0}f(x)g(x)=+\infty.$$
Now assume that $\lim_{x\to x_0}f(x)=+\infty$. Then
$$\forall M>0,\exists\delta>0,\forall x\in D,x\neq x_0,|x-x_0|<\delta\Rightarrow f(x)>M.$$

In particular, for $M = 1$:

$$\exists \delta_3 > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta_3 \Rightarrow f(x) > 1.$$

On the other hand, since $\lim_{x\to x_0} g(x) = +\infty$, we have

$$\forall M > 0, \exists \delta_4 > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta_4 \Rightarrow g(x) > M.$$

Set $\delta = \min(\delta_3, \delta_4)$ then we have $f(x)g(x) > M$. Therefore

$$\forall M > 0, \exists \delta > 0, \forall x \in D, x \neq x_0, |x - x_0| < \delta \Rightarrow f(x)g(x) > M \Leftrightarrow \lim_{x\to x_0} f(x)g(x) = +\infty.$$

■

R Left and right limit versions of this proposition also hold.

R Note that when the limit of f is finite, the inequality is a **strict** inequality. Indeed, if $\lim_{x\to x_0} f(x) = 0$ we have an inderterminate case ($0.\infty$ can be any number).

■ Example 4.6
Determine $\lim_{x\to\pi/2\pm} \tan(x)$.
Solution: we have

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = (\sin(x))\frac{1}{\cos(x)}$$

and

$$\lim_{x\to\pi/2\pm} \sin(x) = 1 > 0.$$

From the previous example, we have $\lim_{x\to\pi/2+} \frac{1}{\cos(x)} = -\infty$ therefore

$$\lim_{x\to\pi/2+} \tan(x) = -\infty.$$

On the other hand, we have we have $\lim_{x\to\pi/2-} \frac{1}{\cos(x)} = +\infty$ therefore

$$\lim_{x\to\pi/2-} \tan(x) = +\infty.$$

Proposition 4.2.3
Assume that $\lim_{x\to x_0} f(x) = L$ (finite) or $\lim_{x\to x_0} f(x) = +\infty$ and $\lim_{x\to x_0} g(x) = +\infty$. Then

$$\lim_{x\to x_0} (f(x) + g(x)) = +\infty.$$

Proof. The proof is similar to the one used for Proposition 3.9.3. Its adaptation is left as an exercise. ■

4.3 Limits at infinity

Definition 4.3.1 — Finite limits at infinity.
We say

- $\lim_{x\to+\infty} f(x) = L$ if and only if

$$\forall\varepsilon > 0, \exists X > 0, \forall x \in \mathbb{R}, x > X \Rightarrow |f(x) - L| < \varepsilon.$$

- $\lim_{x\to-\infty} f(x) = L$ if and only if

$$\forall\varepsilon > 0, \exists X > 0, \forall x \in \mathbb{R}, x < -X \Rightarrow |f(x) - L| < \varepsilon.$$

Example 4.7
Show that $\lim_{x\to+\infty} \frac{2x}{x+3} = 2$
Solution: we have

$$|f(x) - 2| = \left|\frac{2x}{x+3} - 2\right| = \left|\frac{2x - 2(x+3)}{x+3}\right| = \frac{6}{|x+3|} < \frac{6}{|x|}.$$

Since we want to find the limit when $x \to +\infty$, we can consider $x > 0$, thus $\frac{6}{|x|} = \frac{6}{x}$. In order to get $\forall\varepsilon > 0, |f(x) - 2| < \varepsilon$, it is sufficient to choose

$$\frac{6}{x} < \varepsilon \Leftrightarrow x > \frac{6}{\varepsilon}.$$

Therefore, if we set $X = \frac{6}{\varepsilon}$, we get

$$\forall\varepsilon > 0, \exists X > 0, X = \frac{6}{\varepsilon}, \forall x, x > X \Rightarrow |f(x) - 2| < \varepsilon \Leftrightarrow \lim_{x\to\in+\infty} \frac{2x}{x+3} = 2.$$

Definition 4.3.2 — Infinite limits at infinity.
We say

- $\lim_{x\to+\infty} f(x) = +\infty$ if and only if

$$\forall M > 0, \exists X > 0, \forall x \in \mathbb{R}, x > X \Rightarrow f(x) > M.$$

- $\lim_{x\to+\infty} f(x) = -\infty$ if and only if

$$\forall M > 0, \exists X > 0, \forall x \in \mathbb{R}, x > X \Rightarrow f(x) < -M.$$

- $\lim_{x\to-\infty} f(x) = +\infty$ if and only if

$$\forall M > 0, \exists X > 0, \forall x \in \mathbb{R}, x < -X \Rightarrow f(x) > M.$$

- $\lim_{x\to-\infty} f(x) = -\infty$ if and only if

$$\forall M > 0, \exists X > 0, \forall x \in \mathbb{R}, x < -X \Rightarrow f(x) < -M.$$

Example 4.8
Show that $\lim_{x\to+\infty}(3x^2 - 5x) = +\infty$.

Solution: we have

$$3x^2 - 5x = x^2\left(3 - \frac{5}{x}\right).$$

Since we are interested by $x \to +\infty$, we can consider $x > 5$ thus

$$\frac{5}{x} < 1 \Leftrightarrow -\frac{5}{x} > -1 \Leftrightarrow 3 - \frac{5}{x} > 3 - 1 = 2.$$

Then

$$3x^2 - 5x = x^2\left(3 - \frac{5}{x}\right) > 2x^2.$$

In order to have $\forall M > 0, 3x^2 - 5 > M$ it is sufficient to choose $2x^2 > M \Leftrightarrow x^2 > \frac{M}{2} \Leftrightarrow x > \sqrt{\frac{M}{2}}$.
Therefore, keeping in mind that we must have $x > 5$, if we set $X = \max\left(5, \sqrt{\frac{M}{2}}\right)$, we have

$$\forall M > 0, \exists X > 0, X = \max\left(5, \sqrt{\frac{M}{2}}\right), \forall x \in \mathbb{R}, x > X \Rightarrow (3x^2 - 5x) > M$$

$$\Leftrightarrow \lim_{x \to +\infty} (3x^2 - 5x) = +\infty.$$

Theorem 4.3.1
Assume that f is monotone increasing on $[a, +\infty)$.
If $\exists M > 0, \forall x, x \geq a, f(x) < M$ then $\lim_{x \to +\infty} f(x)$ exists (and is finite).
If such M does not exist then $\lim_{x \to +\infty} f(x) = +\infty$.

Proof. Let f be a monotone increasing function on $[a, +\infty)$. Assume $\exists M > 0, \forall x \geq a, f(x) < M$, then

$$L = \sup\{f(x)/x \geq a\}$$

is finite (since M is an upper bound of this set). We will prove that $\lim_{x \to +\infty} f(x) = L$. Indeed, from the definition of the least upper bound, we have

$$\forall \varepsilon > 0, \exists X \in [a, +\infty), L - \varepsilon < f(X) \leq L.$$

Since f is monotone increasing, $\forall x \in [a, +\infty), x > X, L - \varepsilon < f(X) \leq f(x) \leq L < L + \varepsilon$. Therefore, we get

$$\forall \varepsilon > 0, \exists X \in [a, +\infty), \forall x \in [a, +\infty), x > X \Rightarrow |f(x) - L| < \varepsilon \Leftrightarrow \lim_{x \to +\infty} f(x) = L.$$

On the other hand, if the set $\{f(x)/x \geq a\}$ is not bounded above then $\forall M > 0, \exists X \in [a, +\infty), f(X) > M$. Since f is monotone increasing, we have $\forall x \in [a, +\infty), x > X, f(x) \geq f(X) > M$. Therefore

$$\forall M > 0, \exists X \in [a, +\infty), \forall x \in [a, +\infty), x > X \Rightarrow f(x) > M \Leftrightarrow \lim_{x \to +\infty} f(x) = +\infty.$$

■

Theorem 4.3.2 — Cauchy condition for limits at infinity

Let a function $f : [a,+\infty) \to \mathbb{R}$ then $\lim_{x\to+\infty} f(x)$ exists and is finite if and only if

$$\forall \varepsilon > 0, \exists X > 0, \forall b,c \in [a,+\infty), c > b \geq X \Rightarrow |f(c) - f(b)| < \varepsilon.$$

Proof. $\Rightarrow$: Assume that $\lim_{x\to+\infty} f(x) = L$ (finite), i.e

$$\forall \varepsilon > 0, \exists X \in [a,+\infty), \forall x \in [a,+\infty), x > X \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}.$$

Thus for $\forall b,c \in [a,+\infty), c > b \geq X$ we have $|f(c) - L| < \frac{\varepsilon}{2}$ and $|f(b) - L| < \frac{\varepsilon}{2}$. Therefore,

$$\forall b,c \in [a,+\infty), c > b \geq X, |f(c) - f(b)| = |f(c) - L + L - f(b)| \leq |f(c) - L| + |L - f(b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, assume that

$$\forall \varepsilon > 0, \exists X > 0, \forall b,c \in [a,+\infty), c > b \geq X \Rightarrow |f(c) - f(b)| < \varepsilon.$$

Consider the sequence $\{f(n)\}_{n=N_0}^{\infty}$ where $N_0 \geq a$. We have

$$\forall m,n \in \mathbb{N}, m > n \geq N_0 \Rightarrow |f(m) - f(n)| < \varepsilon.$$

Thus $\{f(n)\}$ is a Cauchy sequence, i.e $\exists L \in \mathbb{R}, \lim_{n\to+\infty} f(n) = L$. Now, we will prove that $\lim_{x\to+\infty} f(x) = L$.
From the Cauchy condition, we have

$$\forall x \in [a,+\infty), \forall n \in \mathbb{N}, n > x \geq X \Rightarrow |f(x) - f(n)| < \frac{\varepsilon}{2}.$$

Since $\lim_{n\to+\infty} f(n) = L$, we have $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |f(n) - L| < \frac{\varepsilon}{2}$. Thus

$$|f(x) - L| = |f(x) - f(n) + f(n) - L| \leq |f(x) - f(n)| + |f(n) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\forall \varepsilon > 0, \exists X \in [a,+\infty), \forall x \in [a,+\infty), x > X \Rightarrow |f(x) - L| < \varepsilon \Leftrightarrow \lim_{x\to+\infty} f(x) = L.$$

■

These two theorems are very useful in the study of improper integrals.