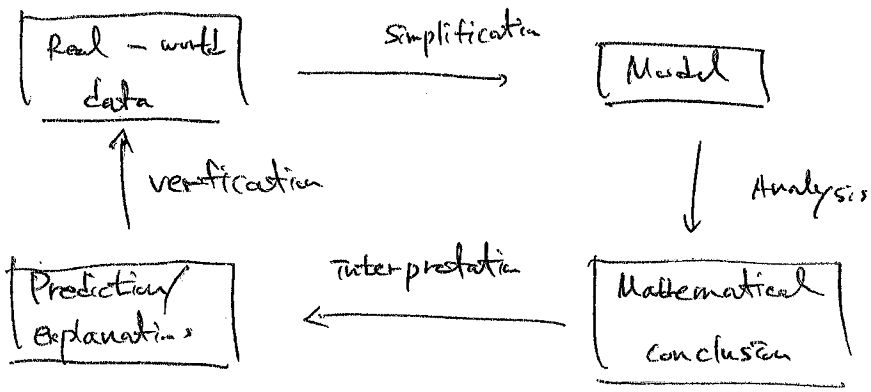
Intro

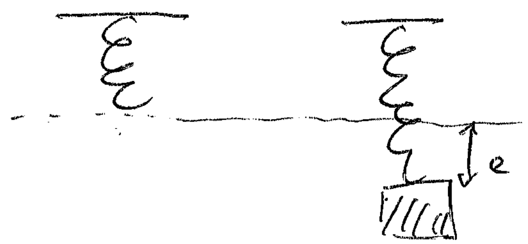


Def Two variables y and x are proportional if

$y = kx$ for some non-zero constant k.

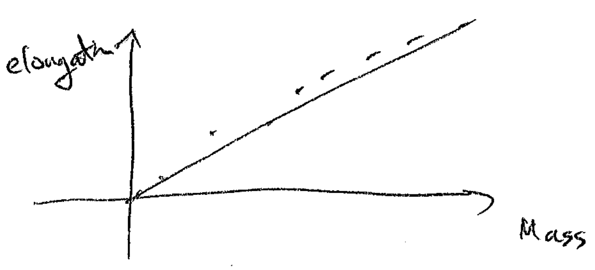
we write $y \propto x$.

(e.g.) Spring-mass system



e : elongation.

we know $e \propto m$ by experiments.



Modeling Change

A powerful paradigm to use in modeling change is

future value = present value + change.

$$b_{n+1} = b_n + 0.01 b_n - 880.87 \quad w/ \quad b_0 = 80{,}000$$

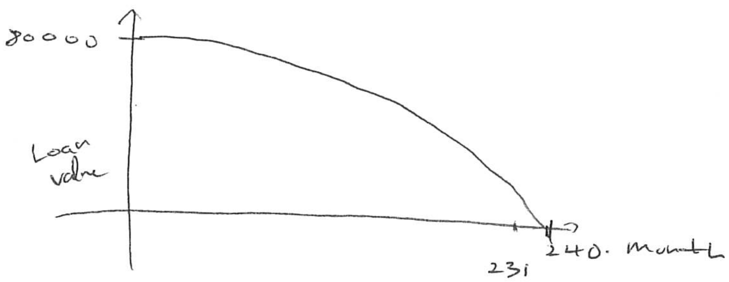
(dynamical system)

where b_n represents the amount owed after n months.

$\Rightarrow b_1 = 80{,}000 + 0.01(80{,}000) - 880.87 = 79919.13$

$b_2 = 79919.13 + 0.01(79919.13) - 880.87 = 79837.45$

$\Rightarrow B = (80{,}000, 79919.13, 79837.45, \ldots)$



(E.g.) A savings certificate initially worth \$1000 that accumulates interest paid each month at 1% per month

Value of the certificate month by month:

$A = (1000, 1010, 1020.10, 1030.30, \ldots$

$\Delta a_0 = a_1 - a_0 = 1010 - 1000 = 10$

$\Delta a_1 = a_2 - a_1 = 1020.10 - 1010 = 10.10$

$\Delta a_2 = a_3 - a_2 = 1030.30 - 1020.10 = 10.20$

$\Delta a_n = a_{n+1} - a_n = 0.01 a_n$

$$\Rightarrow \begin{cases} a_{n+1} = a_n + 0.01 a_n = (1.01) a_n, & n = 0, 1, 2, 3 \\ a_0 = 1000 \end{cases}$$

If $|r|<1$, we consider two cases:

① $0<r<1 \Rightarrow r^k$ approaches 0 as k is larger.

Hence, $a_k = r^k a_0 \longrightarrow 0$ as k is larger.

② $-1<r<0 \Rightarrow a_k = r^k a_0$ alternates in sign as it approaches 0.

< Dynamical system $a_{n+1} = ra_n + b$, r and b are constant >.

Def A number a is called an equilibrium value or fixed point of $a_{n+1} = f(a_n)$ if $a_k = a$ for all $k=1,2,3,\cdots$ when $a_0 = a$. That is, $a_k = a$ is constant solution to the dynamical system.

(E.g.) Digoxin is used in the treatment of heart patients. We consider the decay of digoxin in the bloodstream to prescribe a dosage that keeps the concentration between acceptable level. Suppose we prescribe a daily drug dosage of 0.1 mg and know that half the digoxin remains in the system at the end of each dosage period.

$$\Rightarrow \quad a_{n+1} = 0.5\,a_n + 0.1$$

We consider 3 starting values (initial ~~dose~~ doses)

A: $a_0 = 0.1$

B: $a_0 = 0.2$

C: $a_0 = 0.3$.

For the digoxin example, a_k approaches the equilibrium value 0.2 as k becomes larger. Since $r^k = (0.5)^k$ tends to 0 for large k, we conjecture that

$$a_k = (0.5)^k c + 0.2$$

for some c that depends on the initial condition.

Let's test the conjecture that the form $a_k = r^k c + \frac{b}{1-r}$ solves the system $a_{n+1} = ra_n + b$, $r \neq 1$.

By substitution, $a_{n+1} = ra_n + b \iff r^{n+1} c + \frac{b}{1-r} = r\left(r^n c + \frac{b}{1-r}\right) + b$

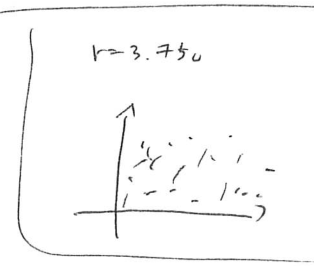
(H.W.)

$$\iff r^{n+1} c + \frac{b}{1-r} = r^{n+1} c + \frac{rb}{1-r} + b$$

$$\iff \frac{b}{1-r} = \frac{rb}{1-r} + b$$

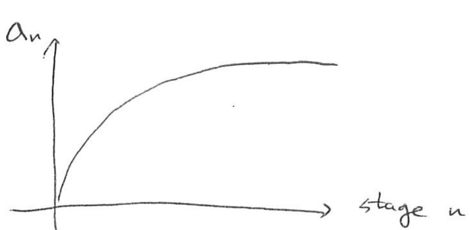
$$\iff b = rb + b(1-r).$$

<u>Thm 3</u> The solution of the dynamical system $a_{n+1} = ra_n + b$, $r \neq 1$ is $a_k = r^k c + \frac{b}{1-r}$ for some c.

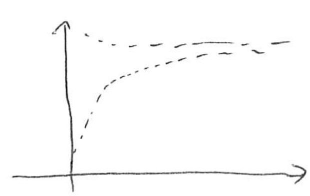


<u>Nonlinear system</u> (H.W.)

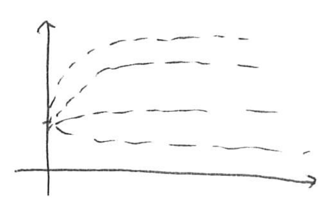
(e.g.) $a_{n+1} = r(1-a_n)a_n$ shows remarkably different behavior w.r.t. r.



$r = 1.546$



$r = 2.75$



$r = 3.325$

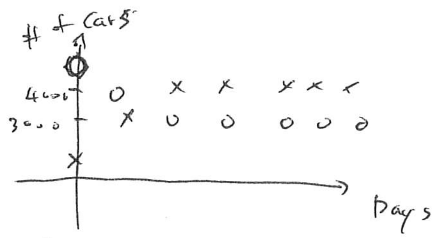
Thus, this system remains at $(S, L) = (3000, 4000)$ if we start there.

② Let's explore the values other than equilibrium.

Consider the system w/

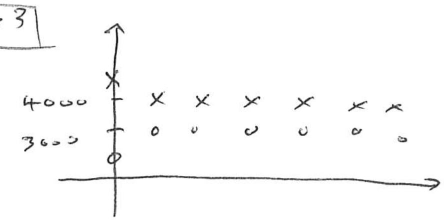
	SD	LA
Case 1	7000	0
Case 2	5000	2000
Case 3	2000	5000
Case 4	0	7000

Case 1

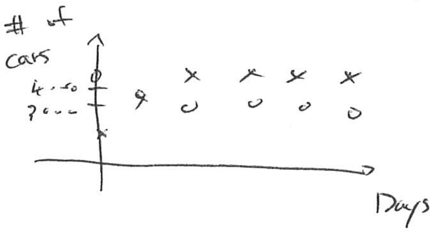


X : LA
O : SD

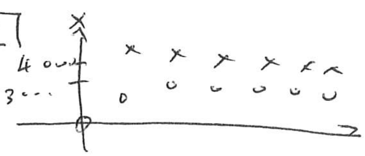
Case 3



Case 2



Case 4



For all 4 cases, the system is very close to the equilibrium values (3000, 4000).

⇒ The equilibrium value is stable and insensitive.

(HW)

(E.g.) Spotted Owls and Hawks

Suppose : (i) Owls competes for survival in a habitat that also supports hawks.

(ii) O_n : the sized of the owl population at the end of day n

H_n : 〃

$\Rightarrow \Delta O_n = k_1 O_n$ and $\Delta H_n = k_2 H_n$, k_1 and k_2 are the constant positive growth rates.

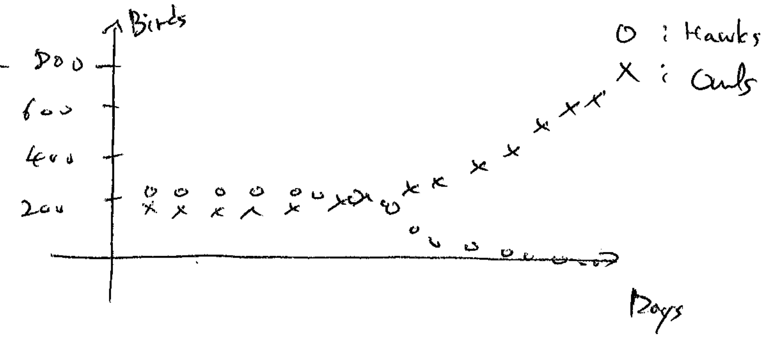
Hence equilibrium $(O, H) = (0, 0)$ and $(O, H) = (150, 200)$

② Let's study the sensitivity of this system.

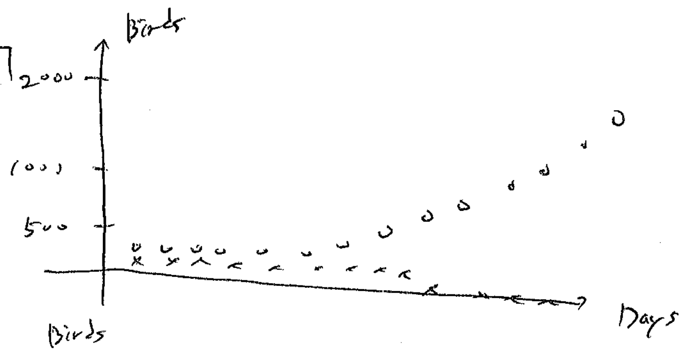
	Owls	Hawk
Case 1	151	199
Case 2	149	201
Case 3	10	10

⇝ I.C. is in the vicinity of (O, H).

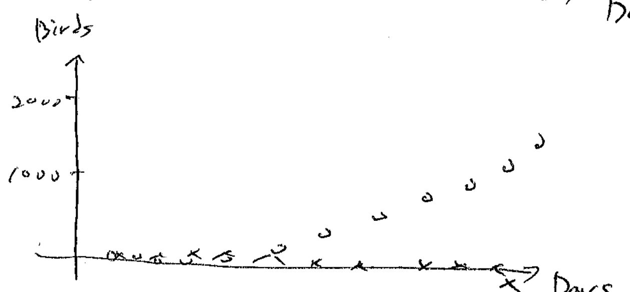
Case 1



Case 2



Case 3



Conclusion

1 Supp. 350 Owls and hawks are to be placed in a habitat modeled. If 150 of ~~the~~ birds are owls, ~~the~~ ~~model~~ ~~the owls~~ the owls will remain at 150 forever.

If 1 owl is removed from ~~the~~ habitat (leaving 149), the model predicts the owl population will die out.

① Equilibrium Values : set (R, D, I), then $R = R_{n+1} = R_n$
$D = D_{n+1} = D_n$
$I = I_{n+1} = I_n$

$$\Rightarrow \begin{cases} -0.25R + 0.2D + 0.4I = 0 \\ 0.05R - 0.4D + 0.2I = 0 \\ 0.2R + 0.2D - 0.6I = 0 \end{cases}$$

There are an infinite # of sol. (How to check?)

$$\begin{pmatrix} -0.25 & 0.2 & 0.4 \\ 0.05 & -0.4 & 0.2 \\ 0.2 & 0.2 & -0.6 \end{pmatrix} \Rightarrow \text{Check the rank.}$$

Let $I=1$, $R = 2.221\cdots$, $D = 0.7777694$ (approximately)

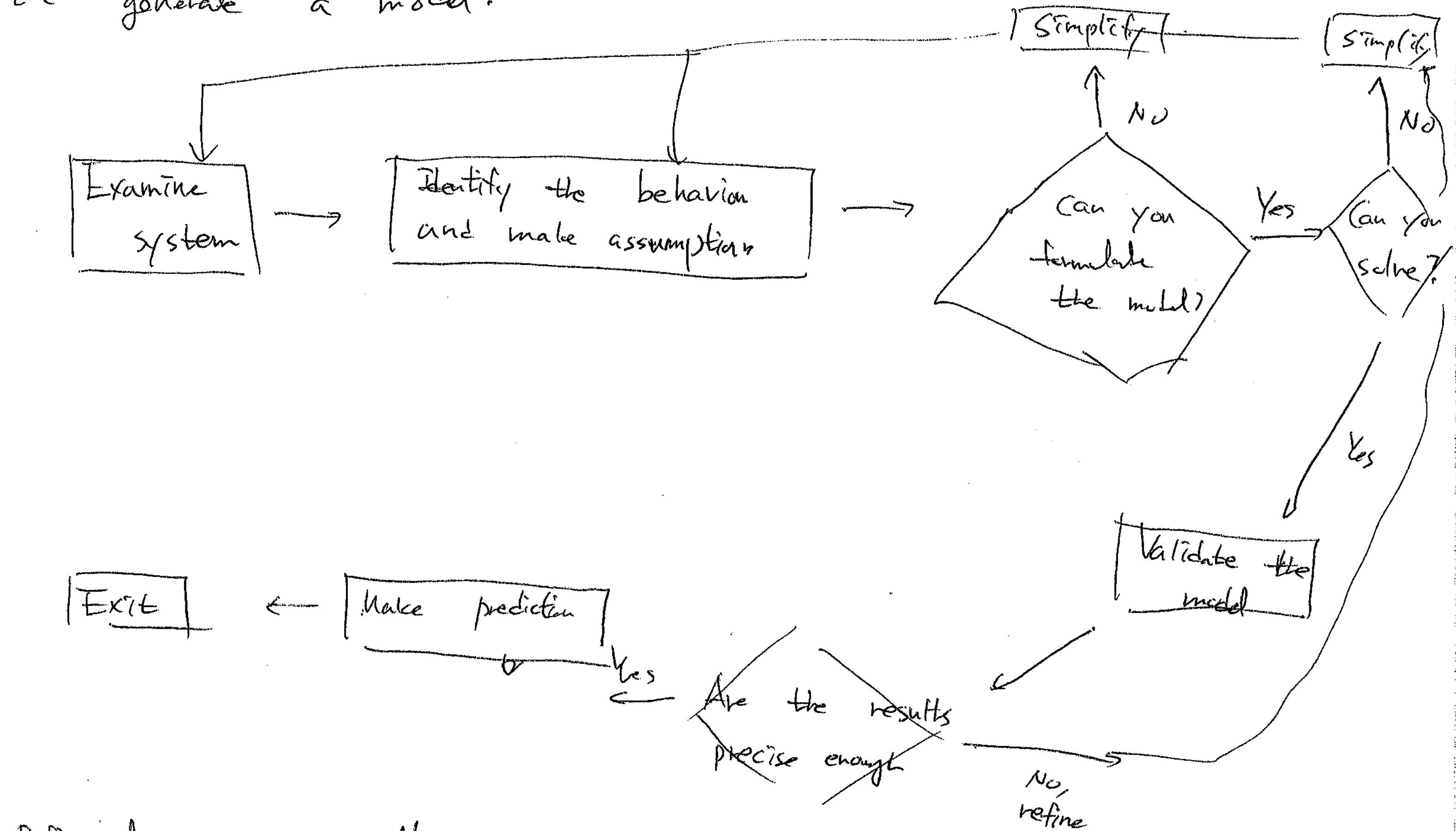
Suppose the system has 399,998 voters. Then, $R = 222,221$, $D = 77,777$, $I = 100,000$ voters should approximately the equil. value.

② Let's ~~check~~ study the system with different I.C.

	R	D	I
1	222,221	77,777	100,000
2	227,221	82,777	90,000
3	100,000	100,000	199,998
4	0	0	399,998

Total voters : 399,998.

Model construction is an iterative process. We begin by examining some system and identifying the particular behavior. Next, we identify the variables and ~~simplyf~~ simplifying assumptions, the generate a model.



Refinement is generally achieved in the opposite way to simplification $\Rightarrow$ we introduce additional variables, assume more sophisticated relationship among the variables, or expand the scope of the problem.

< Modeling using Proportionality >

Recall that $y \propto x \iff y = kx$ for some constant $k > 0$.

Of course if $y \propto x$, then $x \propto y$. (1)

We have other examples:

$$y \propto x^2 \iff y = k_1 x^2 \quad (2)$$

$$y \propto \ln x \iff y = k_2 \ln x \quad (3)$$

$$y \propto e^x \iff y = k_3 e^x \quad (4)$$

For the rules to be the same, at 10 mph both should allow one car length:

$$1 \text{ car length} = \text{distance} = \left(\frac{\text{speed in ft}}{\text{sec}}\right)\cdot(2 \text{ sec})$$

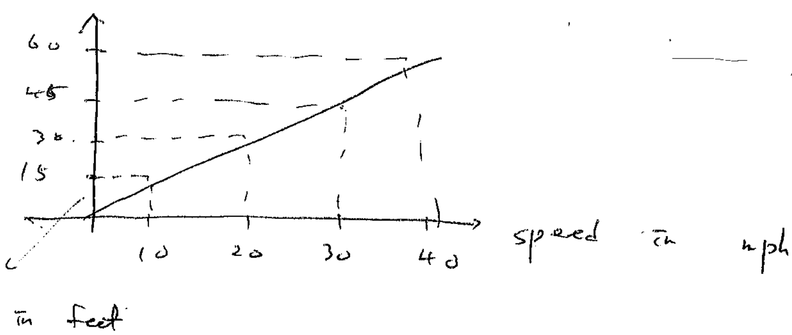
$$= \left(\frac{10 \text{ miles}}{\text{hr}}\right)\left(\frac{5280 \text{ ft}}{\text{mi}}\right)\left(\frac{1 \text{ hr}}{3600 \text{ sec}}\right)\cdot 2 \text{ sec}$$

$$= 29.33 \text{ ft}$$ $\rightsquigarrow$ unreasonable as car length ≈ 15 ft.

So, the rules are not the same.

Let's interpret the one-car-length rule geometrically.

Assume a car length 15 ft.



$\Rightarrow$ proportional $\Rightarrow$ $D = kV$

Hence, $k = \dfrac{15 \text{ ft}}{10 \text{ mph}} = \dfrac{15 \text{ ft}}{52{,}800 \text{ ft}/3600 \text{ sec}} = \dfrac{90}{88}$ sec

We now present a model: total stopping distance

= reaction distance + braking distance

Let's consider the submodels for reaction & braking distance.

(i) Formulation error : Result from the assumptions or from simplification

(e.g.) When we determined a submodel of braking distance, we completely neglect road friction.

(ii) Truncation error : Numerical method error

(e.g.) $\underbrace{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}}_{\text{if we use this only, errors occur.}} - \cdots$

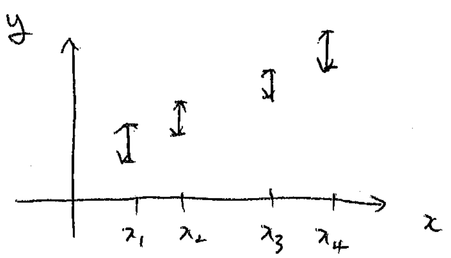
(iii) Round-off errors : Caused by using a finite digit machine

(e.g.) $\frac{1}{3} = \underbrace{0.3333\cdots}\,3\cdots$

(iv) Measurement errors : Caused by ~~impress~~ imprecision in the data collection. (e.g. human errors in recording)

< Fitting models to Data ~~grat~~ graphically >

Suppose we want to fit the model $y = ax + b$ to the data



$\Rightarrow$ each data point is thought of as an interval of confidence.

Q: How we choose the constant a and b to determine the line that best fits the data?

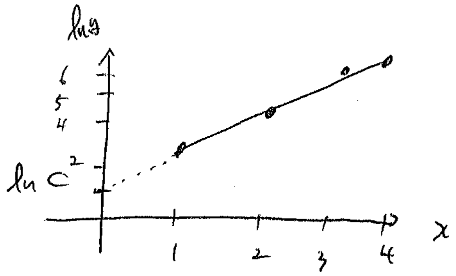
Well, all of them cannot be expected to lie exactly along a single straight line.

In general, there will be some vertical discrepancy. (called absolute ~~deviati~~ deviations)

Now, let's consider an alternative ~~techniq~~ (a useful) technique.

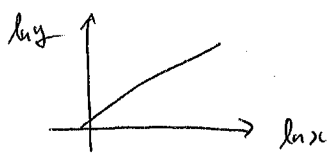
Start with $y = Ce^{x} \Rightarrow \ln y = \ln C + x$.

variables



$\Rightarrow$ (Semilog scale) : It's useful when plotting ~~useful w~~ large amounts of data.

Let's consider $y = x^{a} \Rightarrow \ln y = a \ln x. \Rightarrow \ln y \propto \ln x$

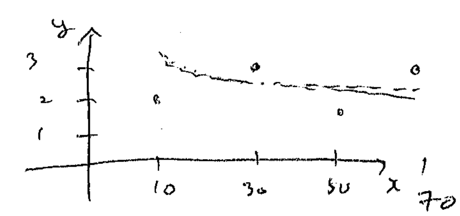


log - log scale.

~~Consider the plot $\ln y$ vs x, and find~~

Note: When transformation of the form $y = \ln x$ are made, the distance concept is distorted.

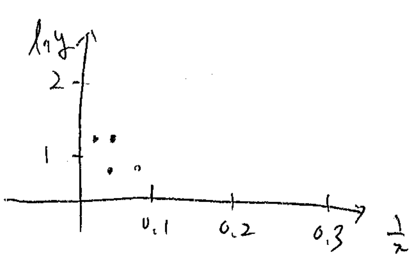
(e.g.) Consider $y = Ce^{1/x}$ and the data plotted : ~~for the follo~~



$y = Ce^{1/x}$ w/. $\ln C = 0.9$.

The data are expected to fit the model $y = Ce^{1/x}$.

Using transformation, $\ln y = \frac{1}{x} + \ln C$



$\Rightarrow$ distance is distorted.

A modeler should be careful when using transformation.

Let's formulate the problem symbolically.

x_1 : true value of the length $\overline{AB}$

x_2 ; 〃 $\overline{BC}$

r_1, r_2, r_3 : discrepancies between the true and measured values

$$\begin{cases} x_1 - 13 = r_1 \\ x_2 - 7 = r_2 \\ x_1 + x_2 - 19 = r_3 \end{cases} \Rightarrow \text{these are called residuals.}$$

Want to minimize the largest of the 3 numbers $|r_1|, |r_2|, |r_3|$.

that is, set $r = \max(|r_1|, |r_2|, |r_3|)$, ~~find~~ ~~to~~ minimize r

subject to

$$|r_1| \le r \Leftrightarrow -r \le r_1 \le r \Leftrightarrow r - r_1 \ge 0 \ \& \ r + r_1 \ge 0$$
$$|r_2| \le r \Leftrightarrow -r \le r_2 \le r \Leftrightarrow r - r_2 \ge 0 \ \& \ r + r_2 \ge 0$$
$$|r_3| \le r \Leftrightarrow -r \le r_3 \le r \Leftrightarrow \qquad 〃$$

Then, we restate this problem as:

Minimize r subject to

$$\begin{cases} r - x_1 + 13 \ge 0 & (r - r_1 \ge 0) \\ r + x_1 - 13 \ge 0 & (r + r_1 \ge 0) \\ r - x_2 + 7 \ge 0 & (r - r_2 \ge 0) \\ r + x_2 - 7 \ge 0 & (r + r_2 \ge 0) \\ r - x_1 - x_2 + 19 \ge 0 & (r - r_3 \ge 0) \\ r + x_1 + x_2 - 19 \ge 0 & (r + r_3 \ge 0) \end{cases}$$

"linear program"
(will discuss this later)

Large linear programs can be solved by computer implementation (simplex method)
→ will study later!

Rmk The Chebyshev criterion is not used often for fitting a curve to a finite collection of data pts. However, whenever minimizing the largest abs. dev. is important, the criterion is considered.

Set d_{max} : the largest of the abs. dev. d_i.

Assume $C_{max} \le d_{max}$, ~~to~~ ~~see~~ ~~interesting~~ for our discussion.

Then, $d_1^2 + d_2^2 + \cdots + d_m^2 \le c_1^2 + \cdots + c_m^2$

Since $c_i \le C_{max}$, $d_1^2 + \cdots + d_m^2 \le m C_{max}^2$

$$\Rightarrow \underbrace{\sqrt{\frac{d_1^2 + \cdots + d_m^2}{m}}}_{:= D} \le C_{max} \qquad \Rightarrow \quad D \le C_{max} \le d_{max}.$$

Hence, if there is considerable difference between D and d_{max}, the modeler should consider applying the cheby. criterion.

<Applying the least square criterion>

① Fitting a straight line

Suppose a model of the form $y = Ax + B$ is ~~expand~~ expected w/ (x_i, y_i), $i = 1, \cdots, m$ to estimate A and B.

Denote the L.S. estimates of $y = Ax + B$ by $y = ax + b$

$$\Rightarrow \text{Minimize} \quad S = \sum_{i=1}^{m} [y_i - f(x_i)]^2 = \sum_{i=1}^{m} (y_i - ax_i - b)^2$$

A necessary cond. for optimality is $\frac{\partial S}{\partial a}$ and $\frac{\partial S}{\partial b} = 0$.

$$\Rightarrow \begin{cases} \dfrac{\partial S}{\partial a} = -2 \displaystyle\sum_{i=1}^{m} (y_i - ax_i - b) x_i & \Rightarrow \; a \displaystyle\sum_{i=1}^{m} x_i^2 + b \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} x_i y_i \\ \dfrac{\partial S}{\partial b} = -2 \displaystyle\sum_{i=1}^{m} (y_i - ax_i - b) = 0 & \Rightarrow \; a \displaystyle\sum_{i=1}^{m} x_i + mb = \sum_{i=1}^{m} y_i \end{cases} \Bigg\} \text{normal eq.}$$

$$\Rightarrow \quad a = \frac{m \sum x_i y_i - \sum x_i \sum y_i}{m \sum x_i^2 - (\sum x_i)^2} \quad ; \text{ slope}$$

$$b = \frac{\sum x_i^2 \sum y_i - \sum x_i y_i \sum x_i}{m \sum x_i^2 - (\sum x_i)^2} \quad ; \text{ intercept}$$

H.W.

3. Transformed L-S fit

In practice, L-S is not easy.

Consider fitting the model $y = Ae^{Bx}$ using L-S criterion.

Call the L-S estimate of $\underbrace{f(x) = ae^{bx}}_{\text{the model}}$.

$\Rightarrow$ minimize $S = \sum_{i=1}^{m} [y_i - f(x_i)]^2 = \sum [y_i - ae^{bx_i}]^2$.

A ~~nec~~ necessary cond. for optimality is $\frac{\partial S}{\partial a} = \frac{\partial S}{\partial b} = 0$.

$\Rightarrow$ not easy to solve ~~nonlin~~ nonlinear system.

Hence, use transformation to approximate the L-S model.

If ~~a~~ ~~a~~ the problem takes a form $Y = AX + B$ in the transformed variables X and Y, the normal eq. can easily fit a line to the transformed variables.

Supp. ~~if~~ we want to fit the power curve $y = Ax^N$ to data.

Denote the estimate of A by α

\+ N by n.

Take $\ln$ on both ~~all~~ sides of $y = \alpha x^n$.

$\Rightarrow \ln y = \ln\alpha + n\ln x$ $\Rightarrow$ straight line for $\ln y$ vs $\ln x$

Using the previous table 1, $m=5$, then slope n and intercept $\ln\alpha$ can be derived

$$n = \frac{5\sum(\ln x_i)(\ln y_i) - (\sum \ln x_i)(\sum \ln y_i)}{5\sum(\ln x_i)^2 - (\sum \ln x_i)^2}$$

As the largest abs. dev. is 0.3476 when $x=2$,

~~Cmax~~ $D = 0.2067 \le c_{max} \le 0.3476 = d_{max}$.

$$\boxed{D = \left(\frac{\sum d_i}{5}\right)^{\frac{1}{2}}}$$

Let's find c_{max} for Chebyshev method! Minimize the largest of the five members $|r_i| = |y_i - y(x_i)|$. Denote our model by $y = a_2 x^2$

Here, Minimize r subject to

$r - r_1 = r - (0.7 - 0.25 a_2) \ge 0$,
$r + r_1 = r + (0.7 - 0.25 a_2) \ge 0$,
$r - r_2 = r - (3.4 - a_2) \ge 0$,
$r + r_2 = r + (3.4 - a_2) \ge 0$,

$r - r_3 = r - (7.2 - 2.25 a_2) \ge 0$
$r + r_3 = r + (7.2 - 2.25 a_2) \ge 0$
$r - r_4 = r - (12.4 - 4 a_2) \ge 0$
$r + r_4 = r + (12.4 - 4 a_2) \ge 0$
$r - r_5 = r - (20.1 - 6.25 a_2) \ge 0$
$r + r_5 = r + (20.1 - 6.25 a_2) \ge 0$

$\Rightarrow$ solution yields $r = 0.28293$ and $a_2 = 3.17073$ (will study later).

Hence, we conclude ~~that~~

	Model	$\sum (y_i - y(x_i))^2$	Max $\lvert y_i - y(x_i)\rvert$
L-S	$y = 3.1869 x^2$	0.2095 ✓	0.3476
Transformed	$y = 3.1368 x^2$	0.3633	0.4950
Chebyshev	$y = 3.17073 x^2$	0.2256	0.2829 ✓

No easy answer about the "best model".

< Discrete optimization modeling >

1. Overview

The basic model is:

$$\text{Optimize } f_j(X) \text{ for } j \text{ in } J \quad \text{subject to} \quad g_i(X) \left\{ \begin{matrix} \ge \\ = \\ \le \end{matrix} \right\} b_i \quad \text{for all } i \text{ in } I.$$

In addition, he has signed contracts to deliver 4 tables and 2 bookcases every week. How to maximize ~~the~~ his profit?,

Maximize $25x_1 + 30x_2$ subject to
$$\begin{cases} 20x_1 + 30x_2 \le 600 & (\text{number}) \\ 5x_1 + 4x_2 \le 4 & (\text{labor}) \\ x_1 \ge 4 & (\text{contract}) \\ x_2 \ge 2 & (\quad 〃 \quad). \end{cases}$$

(E.g.2) (Integer optimization programs)

A space shuttle. There are restrictions on weight & volume capacities.

Supp. there are m different items, each given some numeric value c_j, weight w_j, volume v_j.

Suppose the goal is to maximize the value $\sum_{j=1}^{m} c_j y_j$

where $y_j = \begin{cases} 1, & \text{if item } j \text{ is taken} \\ 0, & \quad 〃 \quad \text{not} \quad 〃 \end{cases}$

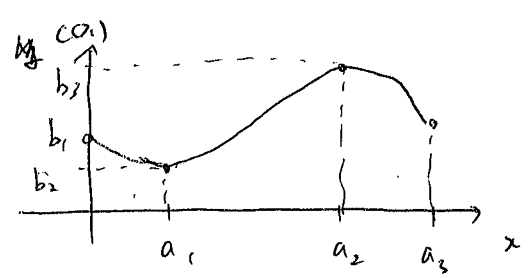
subject to $\sum_{j=1}^{m} v_j y_j \le V$, $\sum_{j=1}^{m} w_j y_j \le W$

where W : weight limit

V : volume limit

(E.g.3) (Approximation by a p.w. linear func.)

Supp. a nonlinear function represent a cost function.



Want to ~~minim~~ find its min. over $0 \le x \le a_3$.

Hence, we have

Minimize $k_1x_1 + k_2x_2 + k_3x_3 + y_1b_1 + y_2b_2 + y_3b_3$ subject to

$0 \le x_1 \le y_1 a_1$

$0 \le x_2 \le y_2 (a_2 - a_1)$ where y_1, y_2 and y_3 equal 0 or 1.

$0 \le x_3 \le y_3 (a_3 - a_2)$

This is called a mixed-integer programming.

2. Linear ~~programming~~ programming I : Geometric solutions.

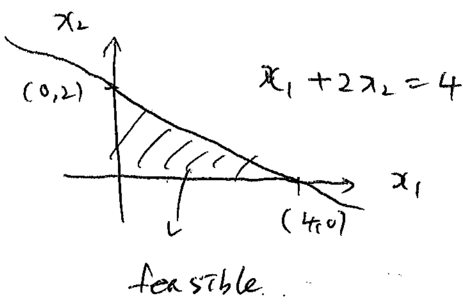
Consider the constraints to gain some insight :

$$\begin{cases} x_1 + 2x_2 \le 4 \\ x_1, x_2 \ge 0 \end{cases}$$

$x_1, x_2 \ge 0 \longrightarrow$ nonnegativity means solutions lie in the 1st quadrant.

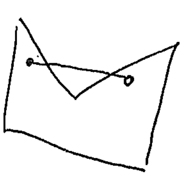
The ineq. $x_1 + 2x_2 \le 4$ divides the first quadrant into two regions.

The feasible region is the half-space in which the constraint is satisfied.

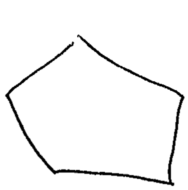


How to check? Pick up one pt and test it.

Notation ① Convex set? If any two of its pts are joined by a straight line segment, then all of these points lie within the set



non-convex



Convex.

x	1	2	3
y	2	5	8

~~The optimize determine~~

Find c to minimize the largest abs. dev

$r_i = |y_i - f(x_i)| \Rightarrow$ linear program

Minimizer r subject to

① $r - r_1 \geq 0$, ③ $r - r_2 \geq 0$, ⑤ $r - r_3 \geq 0$

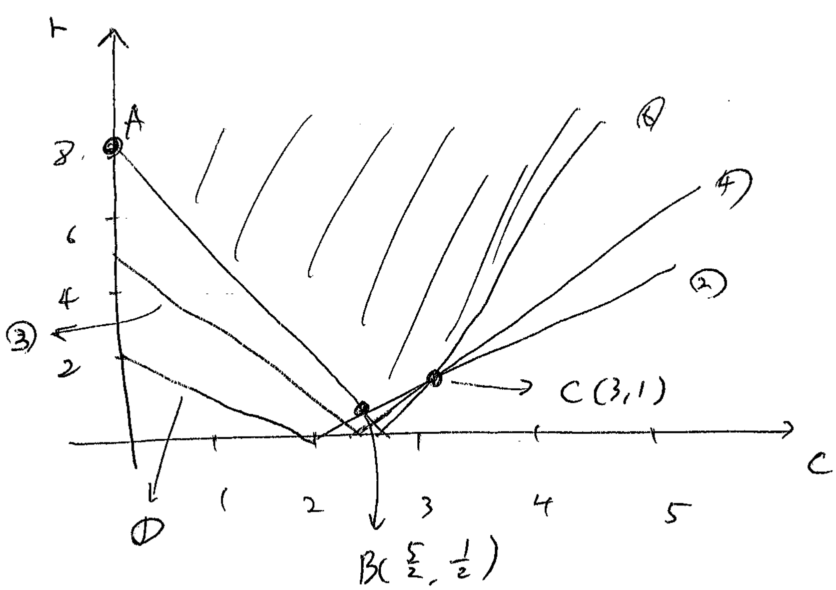
② $r + r_1 \geq 0$, ④ $r + r_2 \geq 0$, ⑥ $r + r_3 \geq 0$

where $r_1 = 2 - f(1) = 2 - c$,

$r_2 = 5 - f(2) = 5 - 2c$,

$r_3 = 8 - f(3) = 8 - 3c$.

$$\boxed{\begin{array}{l} |r_1| \leq r \\ |r_2| \leq r \\ |r_3| \leq r \end{array}}$$



① $r - (2 - c) \geq 0$

② $r + (2 - c) \geq 0$

③ $r - (5 - 2c) \geq 0$

④ $r + (5 - 2c) \geq 0$

⑤ $r - (8 - 3c) \geq 0$

⑥ $r + (8 - 3c) \geq 0$.

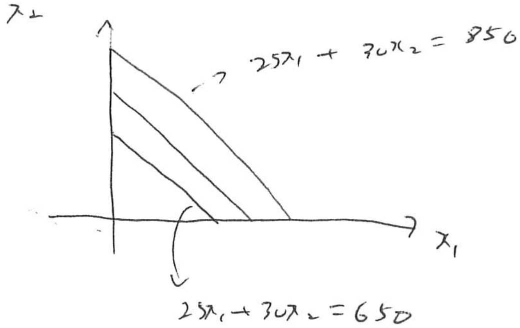
A : intersection of ⑤ and the r-axis

B : " ⑤ and ②

C : " ④ and ② and ⑥

Extreme pt	obj. func. $f(r) = r$
A	8
B	$\frac{1}{2}$ ⟶ smallest.
C	1

Hence, $c = \frac{5}{2}$ is the optimal ~~value~~ value of c.

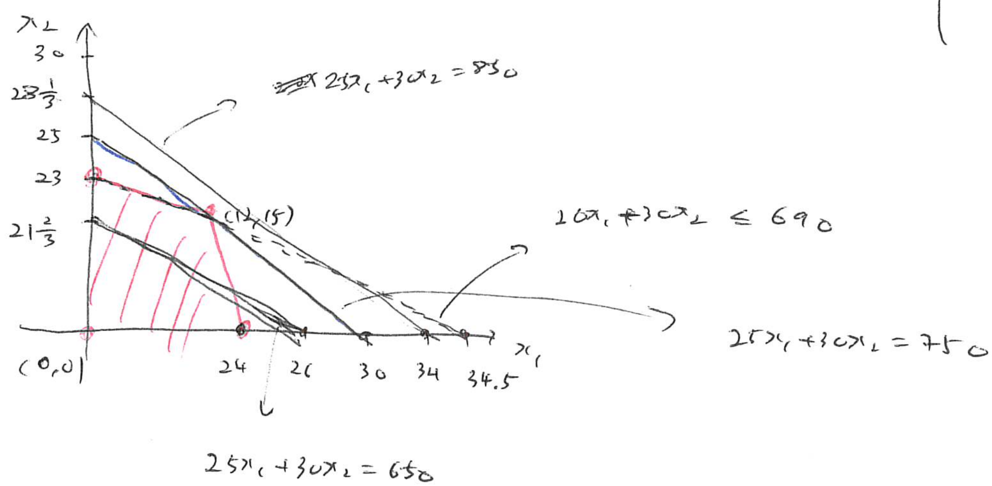


Note that the obj. func has constant value along these line segment. The line segments are called <u>level curves</u> of the obj. func.

As we move in a direction perpendicular to these lines, the obj func either increases or decreases.

Let's reintroduce the constraints:

$$\begin{cases} 20x_1 + 30x_2 \le 690 & \text{(lumber)} \\ 5x_1 + 4x_2 \le 120 & \text{(labor)} \\ x_1, x_2 \ge 0 \end{cases}$$



Notice that the level curve with value 750 is the one that intersects the feasible region at $C(12, 15)$.

Can there be more than one optimal sol?

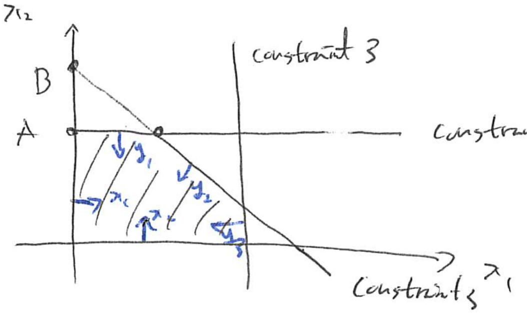
Consider the following slight variation:

Maximize $25x_1 + 30x_2$ subject to
$$\begin{cases} 20x_1 + 30x_2 \le 690 \\ 5x_1 + 6x_2 \le 150 \\ x_1, x_2 \ge 0 \end{cases}$$

3. Linear programming II : algebraic ~~sta~~ solutions

We need to characterize ~~B~~ the intersection points ~~of~~ and extreme pts of the feasible set to implement algebraic method.

Consider the following example : constraint 1 – 3 + $x_1, x_2 \geq 0$



The non-negative variables $y_1, \ldots y_3$ indicated in the figure measure the degree by which a point satisfies each of the constraints 1, 2, 3, respectively. That is, y_i is added to the left side of ineq. constraint i to ~~convert~~ convert it equality. Hence $y_2 = 0$ characterizes those points that lie precisely on constraint 2, and a negative value for y_2 indicates the violation of constraint 2.

Consider the entire set of values $\{x_1, x_2, y_1, y_2, y_3\}$.
If two of them $= 0$, then there is an intersection point.
(could be feasible or infeasible)
↳ negative values for any of the five variables.

Let's illustrate the procedure by solving the carpenter's problem algebraically.

Here, y_1 in our case.

"The ratio represent the value of the entering variable would obtain if the corresponding exiting variable were assigned the value 0".

Thus, the smallest positive value is chosen so as not to drive any variable negative.

For instance, if y_2 were chosen as the exiting variable and assigned the value 0, then x_2 would assume a value 30 as the new dependent variable. However, then y_1 would be negative, indicating that the intersection point (0,30) does not satisfy the 1st constraint.

not feasible

The <u>minimum positive ratio rule</u> obviates enumeration of any infeasible intersection points.

$\Rightarrow$ $\{x_2, y_2, z\}$ new dep. var.

$\{x_1, y_1\} \rightarrow$ new indep. var.

$\}$ y_1 exiting variable

<u>Step 5</u> Pivoting to solve for the new dep. var. value.

Next we derive a new equivalent system by eliminating the entering variable x_2 in all eqs that do not contain the exiting variable y_1.

This is called the <u>pivoting procedure</u>.

Then, we find the values (x_2, y_2, z) when $(x_1, y_1) = (0, 0)$.

At this point, we introduce a table format and summarize ~~as we do not need to know~~ the procedure so far.

① Tableau 0 (original Tableau)

x_1	x_2	y_1	y_2	z	RHS	
20	30	1	0	0	690	$(= y_1)$
5	4	0	1	0	120	$(= y_2)$
-25	-30	0	0	1	0	$(= z)$

Dep. var. $\{y_1, y_2, z\}$
Indep. var. $x_1 = x_2 = 0$
Extrem pt : $x_1, x_2 = 0$
Value of obj func : $Z = 0$

② Optimality test : The entering var. is x_2 (corresponding to -30 in the last row).

③ Feasibility test : compute the ratios for the RHS divided by the coeff. in the column labeled x_2 to determine the minimum positive ratio

entering var.
↓

x_1	x_2	y_1	y_2	z	RHS	Ratio	
20	30	1	0	0	690	23	$(= 690/30)$ → exiting var.
5	4	0	1	0	120	30	$(= 120/4)$
-25	-30	0	0	1	0		

Choose y_1 corresponding to the minimum positive ratio 23 as the exiting variable.

④ Pivot : Divide the row containing the exiting var. (1st row here) by the coefficient of the entering var. in that row (coeff. x_2 here),

⑪ Tableau 2

x_1	x_2	y_1	y_2	Z	RHS
0	1	0.071429	-0.28571	0	15 $(=x_2)$
1	0	-0.057143	0.42857	0	12 $(=x_1)$
0	0	0.714286	2.14286	1	750 $(=Z)$

$-\frac{2}{15} / \frac{7}{3}$ $\qquad$ $1 / \frac{7}{3}$

Dep. var : x_2, x_1, Z

Indep var : $y_1 = y_2 = 0$

~~Expon~~

Extreme pt : $(x_1, x_2) = (12, 15)$

Value of obj func $Z = 750$.

⑫ Optimality test : No negative coeff in the bottom.

$\Rightarrow$ $(12, 15) = (x_1, x_2)$ gives the optimal sol. $Z = 750$.

Note that starting with an initial extreme point, we had to enumerate only two out of 6 intersection points.

$\Rightarrow$ The power of the simplex method is its reduction of the computations

(E.g. 2) Maximize $3x_1 + x_2$ subject to

$$2x_1 + x_2 \le 6$$
$$x_1 + 3x_2 \le 9$$
$$x_1, x_2 \ge 0.$$

pf) The problem in Tableau format is

$$\begin{cases} 2x_1 + x_2 + y_1 = 6 \\ x_1 + 3x_2 + y_2 = 9 \\ -3x_1 - x_2 + Z = 0 \end{cases}, \quad x_1, x_2, y_1, y_2, Z \ge 0.$$

① Tableau 0 (original one)

x_1	x_2	y_1	y_2	Z	RHS
2	1	1	0	0	6 $(=y_1)$
1	3	0	1	0	9 $(=y_2)$
-3	-1	0	0	1	0 $(=Z)$

Dep. var. $\{y_1, y_2, Z\}$

Indep. var $x_1 = x_2 = 0$

Extreme pt $(0, 0) = (x_1, x_2)$

Val. of obj. fun. $Z = 0$.

5. Sensitivity analysis.

Management would like to know if the potential additional profit justifies the cost of another unit of resource. If so, over what range of values for ~~the~~ resources is the analysis valid?

⟹ How sensitive the optimal sol. is to changes in the various constants in the program.

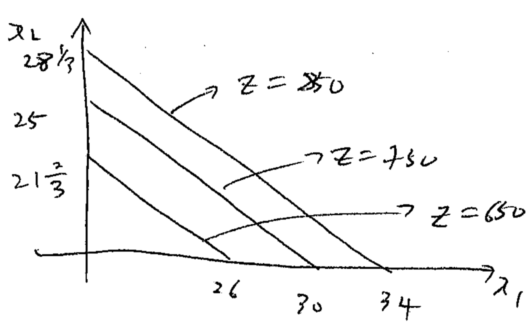
Using the carpenter's problem as an example, we answer the following questions:

1. Over what range of values for the profit per table does the current sol. remain optimal?
2. ~~What's the value of another unit of the 2nd resource?~~ How much will the profit increase if another unit of labor (~~labor~~) is obtained?

<u>Changes in the coeff. of the obj. function</u>

\$ 25 table
\$ 30 bookcase) net profit ⟹ maximize $Z = 25x_1 + 30x_2$.

We can sketch the level curve



⟹ level curves of Z w/ slope $-5/6$.

By previous example, (12, 15) gives optimal sol. $Z = 750$.

Interpretation : Profit per TB > 37.5

$\Rightarrow$ should make only tables (i.e. 24 tables)

Profit per TB < 20

$\Rightarrow$ should make only BCs (i.e. 23 BCs)

Otherwise, (12, 15) is optimal

Note : ① $c \in [20, 37.5]$, the extreme point remains the same but obj func should be different.

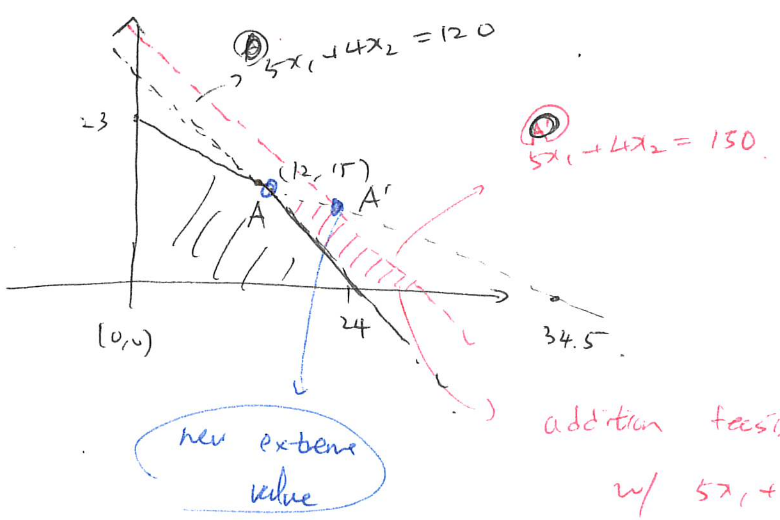
② $c_1 = 20, 37.5$, there are two extreme points

Changes in the r.h.s value (amount of resource available)

What is the effect of increasing the amount of labor?

If b_2 = the units of available labor, then

$5x_1 + 4x_2 \le b_2$.



Consider $b_2 = 120$ and $b_2 = 150$.
the case

~~As~~ b_2 is larger, the graph ~~is~~ moves higher.

As the optimal sol. moves along ~~AA'~~ from A to A', the value of x_1 increases and x_2 decreases.

Goal : Determine how much the obj func val changes as b_2 increases by one unit?

Similarly, we wish to find b_2 at $D(0, 23)$, which is $5 \times 0 + 4 \times 23 = 92$. As b_2 changes, the optimal sol. moves along the lumber constraint as long as $92 \le b_2 \le 172.5$.

How much does the obj fun changes as b_2 increases by 1 unit within $92 \le b_2 \le 172.5$?

(i) Supp. $b_2 = 172.5$. The optimal sol. is the point $E(34.5, 0)$

$\Rightarrow$ obj func. $34.5 \times 25 = 862.5$ at E.

$\Rightarrow$ $862.5 - 750 = 112.5$ unit increases

$$\Rightarrow \frac{862.5 - 750}{172.5 - 120} = 2.14.$$

another way

(ii) If b_2 increases by 1 unit from 120 to 121, then the new extreme pt A' will be the intersection of

$$\begin{aligned} 20x_1 + 30x_2 &= 690 \\ 5x_1 + 4x_2 &= 121 \end{aligned} \Rightarrow A'(12.429, 14.714)$$

$\Rightarrow$ obj func at A' ; 752.14

Thus, the net effect as b_2 increases by 1 unit is to increase the obj func. by 2.14 unit.

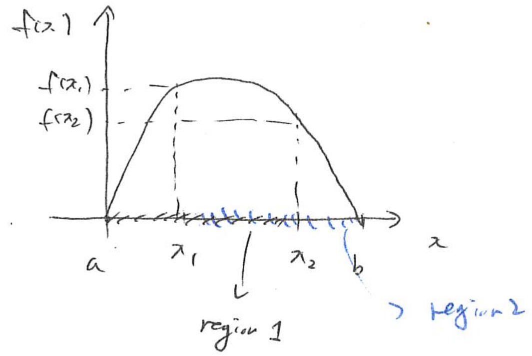
6. Numerical search model.

We want : Maximize a differentiable func. $f(x)$ over $[a, b]$

If $f'(x) = 0 \longrightarrow$ critical point.

$f''(x) \longrightarrow$ characterize the nature of those critical pts.

two test points x_1 and x_2 in $[a, b]$. We then determine the subinterval ~~whether~~ where the optimal sol lies and then use that subinterval to continue the search based on $f(x_1)$ and $f(x_2)$.



There are 3 cases in the max problem

① $f(x_1) < f(x_2)$

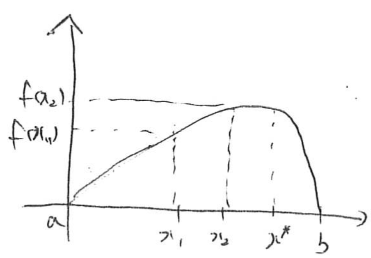
$\Rightarrow$ since $f(x)$ is unimodal, the sol. cannot occur in $[a, x_1]$

$\Rightarrow$ The sol. lie in $(x_2, b]$

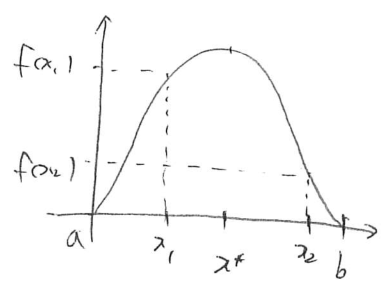
② $f(x_1) > f(x_2)$ $\Rightarrow$ since $f(x)$ is unimodal, the sol. cannot occur in $(x_2, b]$.

$\Rightarrow$ The sol. must lie in $[a, x_2)$

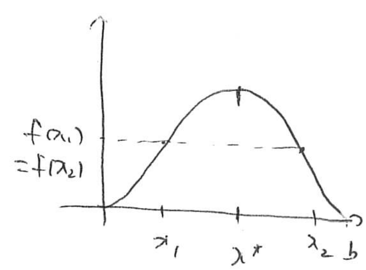
③ $f(x_1) = f(x_2)$ $\Rightarrow$ The sol. must lie in (x_1, x_2).



① $f(x_1) < f(x_2)$



② $f(x_1) > f(x_2)$



③ $f(x_1) = f(x_2)$.

Dichotomous Search Method

Assume $f(x)$ to maximize over $[a, b]$. The dichotomous method computes the mid point $\frac{a+b}{2}$ and ~~move~~ then moves slightly to either side of the midpoint to compute two test points: $\frac{a+b}{2} \pm \varepsilon$, where ε is very small number.

We then find n ; $(0.5)^n = \frac{0.2}{6-(-3)}$ $\Rightarrow$ $n = 5.49$

$\Rightarrow n = 6$ round up.

Then,

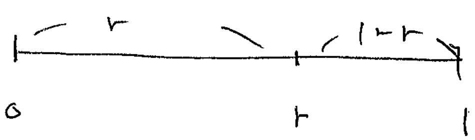
a	b	x_1	x_2	$f(x_1)$	$f(x_2)$
-3	6	1.49	1.51	-5.2001	-5.3001
-3	1.51	-0.755	-0.735	0.94	0.9298
-3	-0.735	-1.8775	-1.8575	0.223	0.2647
-1.8775	〃	-1.3163	-1.2963	0.9	0.9122
-1.3163	〃	-1.0356	-1.0156	0.9987	0.9998
-1.0356	〃	-0.8953	-0.8753	0.9890	0.9845

$$x^* = \frac{-1.0356 - 0.8753}{2} = -0.9555 \Rightarrow f(x^*) = 0.998$$

well, quite close !

Golden Section Search Method

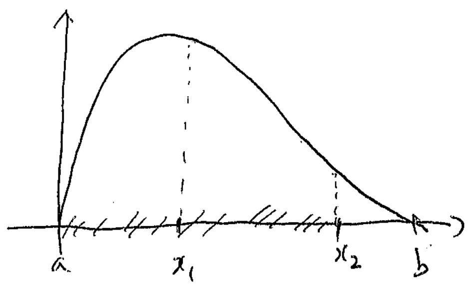
Let's divide the interval $[0,1]$ into two separate subintervals of lengths r and $1-r$. If the length of the whole interval to the length of the longer segment equals the length of the longer segment to the length of the smaller segment, we call this is a "golden ratio".

$\Rightarrow$ 

$$\frac{1}{r} = \frac{r}{1-r} \Rightarrow r^2 + r - 1 = 0$$

$$\Rightarrow r = \frac{\sqrt{5}-1}{2}, \cancel{\frac{-\sqrt{5}-1}{2}}$$

$$\approx 0.618$$



$f(x_1) > f(x_2) \Rightarrow (a, x_2)$ is the new ~~interval~~ interval

(just update one new test point).

(~~Note~~) The number of iterations required to achieve the tolerance length can be found as the integer greater than k,

where $k = \ln\left(\frac{k}{b-a}\right) / \ln[0.618]$. $\left(\text{b/c} \quad (0.618)^k = \frac{\cancel{k} t}{b-a} \Rightarrow k(\ln 0.618) = \ln\left(\frac{t}{b-a}\right) \right).$

(E.g.) ~~Supp~~ ~~we~~ Want to maximize $f(x) = -3x^2 + 21.6x + 1$, $0 \le x \le 25$. with $t = 0.25$. using G.S.M.

(pf) (1) Find

$x_1 = a + 0.382(b-a) \rightarrow x_1 = 9.55$

$x_2 = a + 0.618(b-a) \rightarrow x_2 = 15.45$

$\Rightarrow f(x_1) = -66.3275$, $f(x_2) = -381.3875$.

(2) Since $f(x_1) > f(x_2)$, $(a, x_2) \rightarrow$ new interval $= (0, 15.45)$.

$= (\cancel{x_1}, \cancel{x_2})$ new a b

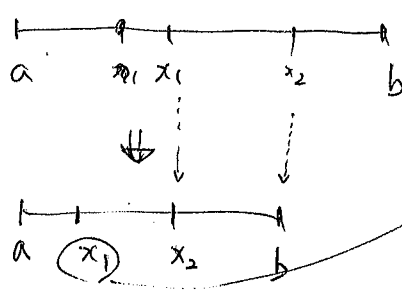
~~$\Rightarrow f(x_2) = -66.3275$ from the previous step~~

(3) Then new $x_2 = 9.55$ which is the previous x_1

$\Rightarrow f(x_2) = -66.3275$.

Let's find the new $x_1 = 0 + 0.382(15.45 - 0) = 5.9017$

$\Rightarrow f(x_1) = 23.9865$.



This is the current situation,

~~Again~~